



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

1960

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196001.01567>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1960. Volume 132, No. 4

MATHEMATICS

P. S. BONDARENKO

**ON THE NUMERICAL CONTINUATION OF
THE SOLUTION OF AN INITIAL-VALUE
PROBLEM FOR ORDINARY DIFFERENTIAL
EQUATIONS**

(Presented by Academician A. A. Dorodnitsyn on 30 I 1960)

This note is devoted to the question of the numerical continuation of the solution of the problem

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0, \quad (k)$$

where $f(t, x)$ is a continuous vector function in some domain D of the $(t, x^{(1)}, \dots, x^{(p)})$ -space, the point $(t_0, x_0) \in D$, on the maximal possible interval $t_0 \leq t \leq T^*$ of the t -line.

1. Suppose that on the interval $t_0 \leq t \leq T$ a set of points is chosen

$$t_j = t_0 + jh_n, \quad h_n = \frac{T - t_0}{n}, \quad j = 0, 1, \dots, n,$$

which in what follows will be called the nodes of this interval. The interval (t_0, T) , on which the nodes are chosen, will be denoted by $(t_0, T)_{h_n}$. Let at each node of the interval $(t_0, T)_{h_n}$ there be defined a p -dimensional vector $c_j = (c_j^{(1)}, \dots, c_j^{(p)})$. Then, obviously, the piecewise-linear function

$$x(t) = x_0 + h_n \sum_{j=0}^{k-1} c_j + (t - t_k)c_k, \quad t_k \leq t \leq t_{k+1}, \quad k = 0, 1, \dots, n-1, \quad (1)$$

is defined on the interval $t_0 \leq t \leq T$.

Remark. If l, m are integers, Σ is the summation symbol, then by definition we set

$$\sum_{j=l}^m c_j = 0 \quad \text{for } l > m.$$

Definition 1. The piecewise-linear function (1) will be called an h_n -approximate solution of problem (k) on the interval $t_0 \leq t \leq T$ if the following conditions are satisfied: 1) for any $n = 1, 2, \dots$,

$$(t, \overset{n}{x}(t)) \in D, \quad t_0 \leq t \leq T;$$

2) for any monotonically decreasing sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers converging to zero,

$$h_n^{-1} \int_{t_k}^{t_{k+1}} \|f(t, \overset{n}{x}(t)) - c_k\| dt \leq \varepsilon_n, \quad k = 0, 1, \dots, n-1.$$

2. It is known ⁽¹⁾ that computational schemes of the Euler, Runge, and Adams type have the form

$$\overset{n}{x}_{k+1} = \overset{n}{x}_k + h_n c(t_k, \overset{n}{x}_k), \quad k = \nu, \dots, n-1, \quad \nu \geq 0. \quad (2)$$

The vectors $c(t_k, \overset{n}{x}_k)$, computed recursively by the numerical method (2), will be called the node coefficients corresponding to this method.

Definition 2. Let $c(t_k, \overset{n}{x}_k)$ be the angular coefficients corresponding to the numerical method (2). The function

$$\overset{n}{x}(t) = \overset{n}{x}_\nu + h_n \sum_{j=\nu}^{k-1} c(t_j, \overset{n}{x}_j) + (t - t_k) c(t_k, \overset{n}{x}_k), \quad (3)$$

$$t_k \leq t \leq t_{k+1}, \quad k = \nu, \dots, n-1,$$

is called the **piecewise-linear function corresponding to the numerical method (2)**.

3. Definition 3. The piecewise-linear function

$$\overset{n_*}{x}^*(t) = \overset{n_*}{x}^*_\nu + h_n \sum_{j=\nu}^{k-1} c(t_j, \overset{n_*}{x}^*_j) + (t - t_k) c(t_k, \overset{n_*}{x}^*_k), \quad (4)$$

$$t_k \leq t \leq t_{k+1}, \quad k = \nu, \dots, n-1,$$

corresponding to the numerical method

$$x_{k+1}^{n_*} = x_k^{n_*} + h_n c(t_k, x_k^{n_*}) - \delta_k, \quad k = \nu, \dots, n-1, \quad (5)$$

is called a **real h_n -approximate solution of problem (k)** on the interval $t_\nu \leq t \leq T$, if the following conditions are satisfied:

- 1) $(t, x^*(t)) \in D$, $t_\nu \leq t \leq T$;
- 2) $\|x_k^{n_*} + h_n c(t_k, x_k^{n_*}) - x_{k+1}^{n_*}\| \leq \delta$.

The vector

$$\delta_k = x_k^{n_*} + h_n c(t_k, x_k^{n_*}) - x_{k+1}^{n_*}$$

is called the **round-off error**, and the numerical method (5) is called a **real numerical method corresponding to the numerical method (2)**.

4. Let the parallelepiped Π

$$|t - t_0| \leq a, \quad \|x - x_0\| \leq b \quad (a, b > 0)$$

belong to the domain D . Put $M = \max_{\Pi} \|f(t, x)\|$, and denote by β the sum of the moduli of the coefficients with which the values of the function $f(t, x)$ enter into the expression for the angular coefficient $c(t_k, x_k)$, corresponding to numerical methods of Adams type ⁽¹⁾.

Lemma 1. *If the function $f(t, x)$ is sufficiently smooth in the domain D , then the piecewise-linear functions $\overset{n}{x}(t)$, corresponding to any of the numerical methods of Euler, Runge, and Adams type, are h_n -approximate solutions of problem (k) on the interval $t_\nu \leq t \leq t_0 + \alpha$, where $\alpha = \min(a, b/M)$ in the case of methods of Euler and Runge type, $\alpha = \min(a, b/\beta M)$ in the case of methods of Adams type, and moreover $(t, \overset{n}{x}(t)) \in \Pi$ for $t_\nu \leq t \leq t_0 + \alpha$.*

Lemma 2. *If condition 2) of Definition 3 is fulfilled, then the piecewise-linear functions $x^*(t)$, corresponding to any of the real numerical methods of Euler, Runge, and Adams type, are real h_n -approximate solutions of problem (k) on the interval $t_\nu \leq t \leq t_0 + \alpha^*$, where $\alpha^* = \min(a, (b - \delta)/M)$ in the case of methods of Euler and Runge type, $\alpha^* = \min(a, (b - \delta)/\beta M)$ in the case of methods of Adams type, and moreover $(t, x^*(t)) \in \Pi$ for $t_\nu \leq t \leq t_0 + \alpha^*$.*

Lemma 3. *If the function $f(t, x)$ is continuous in the domain D and in a neighborhood of zero of the difference $x - y$, $(t, x), (t, y) \in \Pi$, satisfies the Lipschitz condition*

$$\|f(t, x) - f(t, y)\| \leq N(t)\|x - y\|, \quad t_0 \leq t \leq t_0 + \alpha, \quad (6)$$

with a function $N(t)$ summable on the interval $t_0 \leq t \leq t_0 + \alpha$, then on this interval there exists a unique solution $x(t)$ of problem (k), and moreover $(t, x(t)) \in \Pi$ for $t_0 \leq t \leq t_0 + \alpha$ and

$$\lim_{n \rightarrow \infty} x^n(t) = x(t).$$

5. Let $x(t)$ be the solution of problem (k) on the interval $t_0 \leq t \leq t_0 + \alpha$; let $x^{n*}(t)$ be the real h_n -approximate solution of this problem on the interval $t_\nu \leq t \leq t_0 + \alpha^*$.

The difference $u(t) = x(t) - x^{n*}(t)$ is called the **error of the real h_n -approximate solution of problem (k)** on the interval $t_\nu \leq t \leq t_0 + \alpha^*$. Denote by S_n^* the set of nodal points of the interval $(t_\nu, t_0 + \alpha^*)$.

Theorem 1. *If the assumptions of Lemma 3 are satisfied, the function*

$$\gamma(t) = \frac{(f(t, x(t)) - f(t, x^{n*}(t)), x(t) - x^{n*}(t))}{(x(t) - x^{n*}(t), x(t) - x^{n*}(t))}, \quad (7)$$

defined on the interval $t_\nu \leq t \leq t_0 + \alpha^*$, is summable on this interval for any n , and for the error $u(t)$ the estimate

$$\|u(t)\| \leq \|u_\nu\| \exp \left[\int_{t_\nu}^t \gamma(\tau) d\tau \right] + \int_{t_\nu}^t \|\varepsilon_n^x(s)\| \exp \left[\int_s^t \gamma(\tau) d\tau \right] ds, \quad (8)$$

$$t_\nu \leq t \leq t_0 + \alpha^*;$$

$$u_\nu = u(t_\nu), \quad \varepsilon_n^x(t) = f(t, x^{n*}(t)) - \frac{dx^{n*}(t)}{dt}, \quad t \in (t_\nu, t_0 + \alpha^*) - S_n^*. \quad (9)$$

Corollary. *Suppose a function $\vartheta(t)$ has been found, defined on the interval $t_0 \leq t \leq t_0 + \alpha^*$, summable on this interval, and such that the condition*

$$\frac{(f(t, x) - f(t, y), x - y)}{(x - y, x - y)} \leq \vartheta(t) \quad (10)$$

is satisfied for $t_0 \leq t \leq t_0 + \alpha^*$, $\|x - x_0\| \leq b$, $\|y - x_0\| \leq b$. Then for the error $u(t)$ the estimate

$$\|u(t)\| \leq \|u_\nu\| \exp \left[\int_{t_\nu}^t \vartheta(\tau) d\tau \right] + \int_{t_\nu}^t \|\varepsilon_n^*(s)\| \exp \left[\int_s^t \vartheta(\tau) d\tau \right] ds, \quad (11)$$

$$t_\nu \leq t \leq t_0 + \alpha^*.$$

6. The algorithm for continuing a real h_n -approximate solution of problem (k) is as follows. At the i -th continuation step we compute an estimate of

$$\left\| u \left(t_0 + \alpha^* + \sum_{s=1}^{i-1} \alpha_s^* \right) \right\|$$

by means of the majorant (11). Suppose that as a result of this computation we obtain

$$\left\| u \left(t_0 + \alpha^* + \sum_{s=1}^{i-1} \alpha_s^* \right) \right\| \leq r_i.$$

The parallelepiped $\Pi_i \subset D$ is defined by the relations

$$\left| t - t_0 - \alpha^* - \sum_{s=1}^{i-1} \alpha_s^* \right| \leq a_i; \quad \left\| x - x^{n*} \left(t_0 + \alpha^* + \sum_{s=1}^{i-1} \alpha_s^* \right) \right\| \leq b_i;$$

the numbers M_i by the relation

$$M_i = \max_{\Pi_i} \|f(t, x)\|,$$

and the numbers α_i^* by the relation

$$\alpha_i^* = \min \left(a_i, \frac{b_i - r_i - \delta}{\beta M_i} \right),$$

where δ is the bound of the errors of rounding, $\beta = 1$ in the case when the continuation is carried out by methods of Euler and Runge type.

Remark. If the domain D is bounded in at least one of the $p + 1$ coordinate directions, then at each step of the continuation algorithm the parallelepiped Π_i should be constructed so that, in the direction in which the domain D is bounded, its boundary and the boundary of the parallelepiped Π_i are sufficiently close. If the domain D is the whole $(t, x^{(1)}, \dots, x^{(p)})$ -space, then the sequence of parallelepipeds Π_i is constructed with some fixed a and increasing b_i .

Theorem 2. In carrying out the algorithm for continuing a real h_n -approximate solution of problem (k), only the following cases are possible:

- 1) The domain D is bounded at least in the direction of one of the $p + 1$ coordinate axes. At each step of continuation condition (6) is satisfied.
- a) For some finite q it has turned out that $r_q < b_q$, but $r_{q+1} \geq b_{q+1}$. In this case q steps of continuation have led into a sufficiently small neighborhood of the boundary of the domain D . Further continuation is impossible. The solution of problem (k) exists and is unique on the interval

$$t_0 \leq t \leq t_0 + \alpha^* + \sum_{s=1}^q \alpha_s^*.$$

- b) It is established a priori that for every q , $r_q < b_q$. In this case the continuation is infinite. The solution of problem (k) exists and is unique on the half-interval $t_0 \leq t < \infty$.
- 2) The domain D is arbitrary. For some finite q_0 , $r_{q_0} < b_{q_0}$, but on the interval

$$t_0 + \sum_{s=1}^{q_0-1} \alpha_s^* \leq t \leq t_0 + \alpha^* + \sum_{s=1}^{q_0} \alpha_s^*$$

condition (6) is not satisfied. In this case the majorant (11) does not exist, and further continuation is impossible. The solution of problem (k) exists and is unique on the interval

$$t_0 \leq t \leq t_0 + \alpha^* + \sum_{s=1}^{q_0-1} \alpha_s^*.$$

- 3) The domain D is the whole $(t, x^{(1)}, \dots, x^{(p)})$ -space. At each step of continuation condition (6) is satisfied; the numbers r_j , for sufficiently large fixed n_j , increase with each step of continuation. In this case the continuation leads into a neighborhood of a point $t = \tau_0$, at which the solution of problem (k) is unbounded. The solution of the problem exists and is unique on the interval $t_0 \leq t < \tau_0$.
- 4) The domain D is the preceding one. It is established a priori that on any closed interval of the half-interval $t_0 \leq t < \infty$ condition (6) is satisfied and there exists a function $\vartheta(t)$, defined and summable on this interval, such that condition (10) is satisfied. In this case the continuation is infinite. The solution of problem (k) exists and is unique on the half-interval $t_0 \leq t < \infty$.

7. Definition 4. A real h_n -approximate solution of problem (k) is called **stable on the interval** $t_\nu \leq t \leq T$ if it depends continuously on the initial value

$$x^*(t_\nu) = x_\nu^n,$$

and this continuous dependence is uniform with respect to h_n : whatever $\varepsilon > 0$ may be, there exists a $\sigma_\nu > 0$ such that from the inequality

$$\left\| x_\nu^{n^*} - x_\nu^{n^{**}} \right\| \leq \sigma_\nu, \quad (12)$$

there follows the inequality

$$\left\| x_*^n - x^{n^{**}} \right\| = \max_{t_\nu \leq t \leq T} \left\| x^{n^*}(t) - x^{n^{**}}(t) \right\| < \varepsilon, \quad (13)$$

valid for every $n \geq n_0$.

Theorem 3. If the function $f(t, x)$ is sufficiently smooth in the domain D and satisfies condition (6) with $N(t)$ summable on any closed interval (t_0, T^*) contained in the interval of existence of the solution of problem (k), guaranteed by Theorem 2, then the real h_n -approximate solution of this problem, corresponding to any of the real numerical methods of Euler, Runge, and Adams type, is necessarily stable on the interval $t_0 \leq t \leq T^*$.

Kiev State University
named after T. G. Shevchenko

Received
24 I 1960

CITED LITERATURE

1. A. **Kollatz**, *Numerical Methods for Solving Differential Equations*, II, 1953.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.