

SIMULTANEOUS REDUCTION OF A PAIR OF BILINEAR FORMS TO CANONICAL FORM

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Abstract

Full Text

MATHEMATICS

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SIMULTANEOUS REDUCTION OF A PAIR OF BILINEAR FORMS TO CANONICAL FORM

(Presented by Academician A. I. Mal' tsev on 12 I 1960)

A pair of bilinear forms is considered

$$\begin{aligned} A(x, y) &= x' Ay, \\ B(x, y) &= x' By, \end{aligned}$$

where A and B are matrices such that $A' = \alpha A$, $B' = \beta B$ ($\alpha^2 = \beta^2 = 1$), given on an n -dimensional vector space of column vectors over the field of real or complex numbers.

We shall denote a bilinear form by the matrix that corresponds to it in the given coordinate system.

Put $\varepsilon = \alpha\beta$ and introduce the following pairs of forms:

$$\left(\left[\begin{array}{ccc} \overbrace{\rho} & 0 & \overbrace{\rho} \\ & \alpha & \\ 0 & -0 & 0 \\ & 1 & 0 \end{array} \right], \left[\begin{array}{ccc} \overbrace{2\rho+1} & & \\ & \beta & 0 \\ 0 & 1 & 0 \end{array} \right] \right) \quad (\rho \geq 0)_\rho \quad \text{(I)}$$

$$\left(\left[\begin{array}{ccc} & & \pm 1 \\ & \ddots & \pm \varepsilon \\ \pm \varepsilon^2 & & \\ & \ddots & \\ \pm \varepsilon^{\rho-1} & & \end{array} \right], \left[\begin{array}{ccc} & & 0 \\ & & 0 \pm \varepsilon \\ 0 \pm \varepsilon^2 & & \\ & & 0 \pm \varepsilon^{\rho-1} \end{array} \right] \right) \quad (\rho > 0; \alpha\varepsilon^{\rho-1} = 1)_\rho \quad (\pm\text{II})$$

$$\left(\left[\begin{array}{ccc} \overbrace{\rho} & \overbrace{\rho} & \\ \rho & \alpha & \\ & & 1 \end{array} \right], \left[\begin{array}{ccc} \overbrace{\rho} & \overbrace{\rho} & \\ & 0\beta & 0 \\ 0 & & \beta \\ & & 1 \end{array} \right] \right) \quad (\rho > 0; \alpha\varepsilon^{\rho-1} = -1)_\rho \quad \text{(III)}$$

$$\left(\left[\begin{array}{ccc} 0 & & \\ & 0 \pm \varepsilon^2 & \\ & \vdots & \\ 0 \pm \varepsilon^{\rho-1} & & \end{array} \right], \left[\begin{array}{ccc} \pm 1 & & \\ & \pm \varepsilon^2 & \\ & \vdots & \\ \pm \varepsilon^{\rho-1} & & \end{array} \right] \right) \quad (\rho > 0; \beta \varepsilon^{\rho-1} = 1)_\rho$$

(IV)

$$\left(\left[\begin{array}{ccc} \overbrace{\rho} & \overbrace{\rho} & \\ \rho & 0\alpha & \\ 0 & \alpha & 1 \end{array} \right], \left[\begin{array}{ccc} \overbrace{\rho} & \overbrace{\rho} & \\ \rho & \beta & \beta \end{array} \right] \right) \quad (\rho = 0; \beta \varepsilon^{\rho-1} = -1)_\rho \quad (V)$$

$$\left(\overbrace{\begin{array}{ccc} & & \\ & \pm 1 & \\ \pm 1 & & \end{array}}^p, \overbrace{\begin{array}{ccc} & & \\ & \pm a & \\ \pm a \pm 1 & & \pm 1 \end{array}}^p \right) \quad (p > 0; a = \beta = 1) \quad (VI_p(a))$$

} \end{equation}

$$\left(\overbrace{\begin{array}{ccc} & & \\ & a & \\ \cdot & a & \end{array}}^p, \overbrace{\begin{array}{ccc} & & \\ \beta a & \beta & \\ \beta a & \beta & \end{array}}^p \right) \quad (p > 0; \alpha \text{ and } \beta \text{ are not both equal to } 1)$$

(VII_p(a))

\$} \end{equation}

$$\left(\overbrace{\begin{array}{ccc} & & \\ & 1 & \\ 1 & & \end{array}}^p, \overbrace{\begin{array}{ccc} & & \\ a & 1 & \\ b & & \end{array}}^p, \overbrace{\begin{array}{ccc} & & \\ & b & \\ b & & \end{array}}^p \right) \quad (p > 0; \omega = a+ib; \alpha = \beta = 1)$$

(VIII_p(ω))

ω}\$} \end{equation}

where p is a nonnegative integer, p_i is a positive integer ($i = 1, \dots, 6$), a is real or complex $\neq 0$.

The uniqueness of the decomposition into a direct sum means that the number of pairs of each type (from (2) or from (3)) entering into the decomposition of the given pair is an invariant of this pair. We note that this number is completely determined by the system of elementary divisors (finite and infinite) and minimal indices of the pencil $A + \lambda B$ ⁽¹⁾ in the case when K is the field of complex numbers (and also in the case when A and B are real skew-symmetric matrices).

The proof is based on the consideration of vector systems $x_1, \dots, x_p, y_1, \dots, y_p$, whose vectors are related by the relations:

$$(B - aA)x_k = Ax_{k-1}, \quad (B - \varepsilon aA)y_k = \varepsilon Ay_{k-1}, \quad y'_1 Ax_p \neq 0$$

$$(k = 1, \dots, p; x_0 = y_0 = 0; a \text{ is a root of the equation } |B - \lambda A| = 0),$$

and of systems $x_1, \dots, x_p, y_0, y_1, \dots, y_p$, whose vectors are related by the relations:

$$Ay_0 = 0, \quad Ay_k = By_{k-1} \quad (k = 1, \dots, p), \quad By_p = 0;$$

$$y'_p Ax_1 = 1, \quad Ax_k = \varepsilon Bx_{k-1} \quad (k = 2, \dots, p).$$

An analogous result holds for a pair of forms:

$$\begin{aligned} A(x, y) &= x' Ay, \\ H(x, y) &= x' H \bar{y}, \end{aligned} \tag{4}$$

where A is a symmetric or skew-symmetric matrix ($A' = \alpha A$, $\alpha = \pm 1$), and H is a Hermitian matrix, defined on an n -dimensional vector space over the field of complex numbers.

Proposition 2. *Each pair of forms (4) can be represented uniquely (up to the order of the summands) as a direct sum of pairs of forms, each of which is one of the following:*

$$(I)_{p_1}, \quad (II)_{p_1}, \quad -(II)_{2p_2}, \quad (III)_{2p_3-1}, \quad (IV)_{p_4}, \quad -(IV)_{2p_5-1}, \quad (V)_{2p_6},$$

$$(VI)_{p_7}(a), \quad -(VI)_{p_8}(a), \quad (VII)_{p_9}(a), \quad (VIII)_{p_{10}}(\omega), \quad (IX)_{p_{11}}(\omega),$$

where in each pair $\beta = 1$ is set (the second matrix is assumed Hermitian); p is a nonnegative integer, p_i are positive integers ($i = 1, \dots, 11$), a is real and $\neq 0$, and ω is complex.

The classification of pairs of forms (1) presented above in the cases when $\alpha = \beta = 1$ and $\alpha = 1, \beta = -1$, and K is the field of complex numbers, coincides with the known ones ^(1, 2).

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named after V. I. Ulyanov-Lenin

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REFERENCES

1. F. R. Gantmakher, *The Theory of Matrices*, Moscow, 1954, ch. XII, pp. 300–320.
2. W. Hodge, D. Pedoe, *Methods of Algebraic Geometry*, 1, IL, 1954, ch. IX, pp. 416–433.

Note: Figure translations are in progress. See original paper for figures.

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