V. A. SOLONNIKOV

1960

SovietRxiv

Abstract

Full Text

V. A. SOLONNIKOV

ON SOME PROPERTIES OF SPACES $\mathfrak{W}_p^l$ OF FRACTIONAL ORDER

(Presented by Academician V. I. Smirnov on 21 III 1960)

Let us define the space $\mathfrak{W}_p^l(E_n)$ as the closure of smooth finite functions in the space E_n of functions in the norm

$$\|f\|_{\mathfrak{W}p^l(E_n)} = \sum_{i,j=1}^{n} \left(\int_0^\infty \frac{dh}{h^{1+p\lambda}} \int_{E_n} \left| D_i^{\bar{l}} f(x_1 \dots x_j + h \dots x_n) + D_i^{\bar{l}} f(x_1 \dots x_j - h \dots x_n) - 2D_i^{\bar{l}} f(x_1 \dots x_n) \right|^p dx \right)^{1/p}$$

where $l = \bar{l} + \lambda,\ 0 < \lambda \leqslant 1,\ D_i^{\bar{l}} = \dfrac{\partial^{\bar{l}}}{\partial x_i^{\bar{l}}}$. For $\lambda < 1$ this norm is equivalent to the following one:

$$\sum_{i,j=1}^{n} \left(\int_0^\infty \frac{dh}{h^{1+p\lambda}} \int_{E_n} \left| D_i^{\bar{l}} f(x_1 \dots x_j + h \dots x_n) - D_i^{\bar{l}} f(x_1 \dots x_n) \right|^p dx \right)^{1/p}.$$

This fact follows from the identity

$$f(x+h) - f(x) = {}^1\!/_2[f(x+h) - f(x-h)] + {}^1\!/_2[f(x+h) + f(x-h) - 2f(x)].$$

Similar spaces were considered in the works ([1−7]). The spaces $\mathfrak{W}_p^l$ differ from the spaces W_p^l, considered in ([4,7]), only for integer l; moreover, by means of Parseval' s equality it is easy to show that also for integer l the norm $\mathfrak{W}_2^l$ is equivalent to the norm W_2^l on the set of functions finite in some bounded domain. As shown in ([9]), for $p \neq 2$ the norms W_p^l and $\mathfrak{W}_p^l$ are not equivalent.

It can be shown that on the set of functions finite in some bounded domain the norm $\mathfrak{W}_p^l$ is equivalent to the norm $B_{p,p}^l$, introduced in ([6]).

In the present paper we give a new, and, as it seems to us, simpler and more transparent method of proving a number of known facts, and also introduce several new theorems.

Theorem 1. If $f \in \mathfrak{W}_p^l(E_n)$, $m < n$,

$$\sum_{i=1}^{n-m} r_i < l - \frac{n-m}{p},$$

where the r_i are nonnegative integers, then

$$D_{m+1}^{r_1} D_{m+2}^{r_2} \dots D_n^{r_{n-m}} f \in \mathfrak{W}_p^{l-\Sigma r_i - \frac{n-m}{p}}(E_m)$$

for any fixed $x_{m+1}, x_{m+2}, \dots, x_n$, and

$$\left\| D_{m+1}^{r_1} D_{m+2}^{r_2} \dots D_n^{r_{n-m}} f \right\|_{\mathfrak{W}p^{l-\Sigma r_i - \frac{n-m}{p}}(E_m)} \leqslant C \|f\|_{\mathfrak{W}p^l(E_n)}.$$

Theorem 2. If functions $\varphi^i_{r_1^i r_2^i \dots r_{n-m}^i}(x_1 \dots x_m) \in \mathfrak{W}_p^{l-s_i-\frac{n-m}{p}}(E_m)$ are given, where

$$s_i = \sum_{k=1}^{n-m} r_k^i < l - \frac{n-m}{p},$$

then one can construct a function $f \in \mathfrak{W}_p^l(E_n)$ such that

$$\varphi^i_{r_1^i r_2^i \dots r_{n-m}^i}(x_1 \dots x_m) = D_{m+1}^{r_1^i} D_{m+2}^{r_2^i} \dots D_n^{r_{n-m}{}^i} f \Big|_{x_{m+k}=0,\ k=1,\dots,n-m};$$

$$\|f\|_{\mathfrak{W}_p^l(E_n)} \le C \sum_{s_i} \sum_{r_R} \left\| \varphi^i_{r_1^i r_2^i \dots r_{n-m}^i} \right\|_{\mathfrak{W}_p^{l-s_i-\frac{n-m}{p}}(E_m)},$$

where C does not depend on $\varphi^i_{r_1^i r_2^i \dots r_{n-m}^i}$.

Theorem 3. If $f \in \mathfrak{W}_p^l(E_n)$, $l_1 < l$, then $f \in \mathfrak{W}_{p_1}^{l_1}(E_n)$, where

$$l - \frac{n}{p} = l_1 - \frac{n}{p_1}, \qquad \|f\|_{\mathfrak{W}_{p_1}^{l_1}} \le C \|f\|_{\mathfrak{W}_p^l}.$$

Theorem 4. If $f \in \mathfrak{W}_p^l(E_n)$, $lp < n$, then $f \in L_q(E_n)$, where

$$q = \frac{np}{n - lp},$$

and

$$\|f\|_{L_q} \le C \|f\|_{\mathfrak{W}_p^l}.$$

Theorem 5. If $f \in \mathfrak{W}_p^l(E_n)$, $pl > n$, $p(l-1) < n$, then $f \in \mathrm{Lip}_\alpha$, i.e.

$$\frac{|f(x+h) - f(x)|}{|h|^\alpha} \xrightarrow[|h|\to 0]{} 0, \qquad \alpha = l - \frac{n}{p}.$$

Theorem 6. If $f \in \mathfrak{W}_p^l(E_n)$, $pl > n$, $p(l-1) = n$, then the function f satisfies Zygmund' s "condition λ^*," i.e.

$$\frac{|f(x+h)+f(x-h)-2f(x)|}{|h|} \xrightarrow[|h|\to 0]{} 0.$$

The proofs of the theorems are comparatively simple. They are based on representations of the type of the equality

$$f(x) = \frac{1}{h}\int_x^{x+h} f(\xi)\,d\xi - \int_x^{x+h} \frac{dt}{(t-x)^2}\int_x^t [f(t)-f(\xi)]\,d\xi, \tag{1}$$

obtained by V. P. Il' in ([8]) and valid for any h.

Let us illustrate our method of proof on two characteristic examples. It is not difficult to show that the proof of Theorem 1 can be reduced to proving the fact that from $f \in \mathfrak{W}_p^l(E_2)$ it follows that $f \in \mathfrak{W}_p^{l-1/p}(E_1)$. Thus, let $f \in \mathfrak{W}_p^l(E_2)$. We shall show that $f \in \mathfrak{W}_p^{l-1/p}(E_1)$. We restrict ourselves to the case $1/p < l < 1$. Following Gagliardo ([2]), we estimate the $\mathfrak{W}_p^{l-1/p}$-norm of the function $f(x)$ on the straight line $x_1 = x_2$, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(t,t)-f(\tau,\tau)|^p}{|t-\tau|^{lp}}\,d\tau\,dt.$$

We have

$$f(t,t)-f(\tau,\tau) = [f(t,t)-f(t,\tau)] - [f(t,\tau)-f(\tau,\tau)].$$

Let $t \geq \tau$. Consider, for example, the second term. According to (1),

$$f(t,\tau)-f(\tau,\tau) = \frac{1}{t-\tau}\int_\tau^t [f(t,\tau)-f(\xi,\tau)]\,d\xi + \int_\tau^t \frac{d\xi}{(\xi-\tau)^2}\int_\tau^\xi [f(\xi,\tau)-f(\eta,\tau)]\,d\eta \equiv F_1(t,\tau)+F_2(t,\tau).$$

Applying Hölder' s inequality, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} \frac{|F_1|^p}{(t-\tau)^{lp}}\,d\tau &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} \frac{d\tau}{(t-\tau)^{p(l+1)}}\left|\int_\tau^t \frac{f(t,\tau)-f(\xi,\tau)}{(t-\xi)^{l+1/p}}(t-\xi)^{l+1/p}\,d\xi\right|^p \\ &\leq \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} \frac{d\tau}{(t-\tau)^{p(l+1)}}\int_\tau^t \frac{|f(t,\tau)-f(\xi,\tau)|^p}{(t-\xi)^{1+pl}}\,d\xi\left[\int_\tau^t (t-\xi)^{\frac{1+pl}{p-1}}\,d\xi\right]^{p-1} \\ &\leq C\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} d\tau \int_\tau^t \frac{|f(t,\tau)-f(\xi,\tau)|^p}{(t-\xi)^{1+pl}}\,d\xi \leq C\|f\|^p_{\mathfrak{W}_p^l(E_2)}.\end{aligned}$$

We estimate the second term as follows:

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} \frac{|F_2|^p}{(t-\tau)^{lp}}\, d\tau = \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} \frac{d\tau}{(t-\tau)^{lp}} \left| \int_{\tau}^{t} \frac{d\xi}{(\xi-\tau)^2} \int_{\tau}^{\xi} [f(\xi,\tau) - f(\eta,\tau)]\, d\eta \right|^p$$
$$= \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} \frac{d\tau}{(t-\tau)^{lp}} \left| \int_{\tau}^{t} \frac{d\xi}{(\xi-\tau)^{2(1/p+1/p')}} \right.$$
$$\left. \times \int_{\tau}^{\xi} \frac{f(\xi,\tau) - f(\eta,\tau)}{(\xi-\eta)^{l+1/p}} (\xi-\eta)^{(l+1/p)(1/p+1/p')} \left[\frac{(\xi-\tau)^{2/p}}{(t-\tau)^{1/p}} \right]^{1/p'-1/p'} d\eta \right|^p$$
$$\le \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} \frac{d\tau}{(t-\tau)^{lp}} \int_{\tau}^{t} \frac{d\xi}{(\xi-\tau)^2} \int_{\tau}^{\xi} \frac{|f(\xi,\tau) - f(\eta,\tau)|^p}{(\xi-\eta)^{1+pl}} (\xi-\eta)^{l+1/p} \left[\frac{(\xi-\tau)^{2/p}}{(t-\tau)^{1/p}} \right]^{p-1}$$
$$\times \left\{ \int_{\tau}^{t} \frac{d\xi}{(\xi-\tau)^2} \int_{\tau}^{\xi} (\xi-\eta)^{l+1/p} \left[\frac{(\xi-\tau)^{2/p}}{(t-\tau)^{1/p}} \right]^{-1} d\eta \right\}^{p-1}$$
$$= C_1 \int_{-\infty}^{\infty} dt \int_{\tau}^{\infty} \frac{dt}{(t-\tau)^{l+1/p'}} \int_{\tau}^{t} \frac{d\xi}{(\xi-\tau)^{2/p}} \int_{\tau}^{\xi} \frac{|f(\xi,\tau) - f(\eta,\tau)|^p}{(\xi-\eta)^{1+pl}} (\xi-\eta)^{l+1/p}\, d\eta$$
$$= C_1 \int_{-\infty}^{\infty} d\tau \int_{\tau}^{\infty} d\xi \int_{\tau}^{\xi} \frac{|f(\xi,\tau) - f(\eta,\tau)|^p}{(\xi-\eta)^{1+pl}} \frac{(\xi-\eta)^{l+1/p}}{(\xi-\tau)^{2/p}}\, d\eta \int_{\xi}^{\infty} \frac{dt}{(t-\tau)^{l+1/p'}}$$
$$\le C_2 \int_{-\infty}^{\infty} d\tau \int_{\tau}^{\infty} d\xi \int_{\tau}^{\xi} \frac{|f(\xi,\tau) - f(\eta,\tau)|^p}{(\xi-\eta)^{1+pl}}\, d\eta \le C\|f\|^p_{\mathfrak{W}^l_p(E_2)}.$$

The remaining estimates are obtained analogously.

We now show how, with the aid of (1), one can prove Theorem 5 for the case $n = 1,\ l < 1$.

On the basis of (1) one may write

$$f(x+h) = \frac{1}{h} \int_{x}^{x+h} f(\xi)\, d\xi + \int_{x}^{x+h} \frac{dt}{(x+h-t)^2} \int_{t}^{x+h} [f(\xi) - f(t)]\, d\xi,$$

so that

$$\frac{f(x+h) - f(x)}{h^{l-1/p}} \frac{1}{h^{l-1/p}} \int_{x}^{x+h} \frac{dt}{(x+h-t)^2} \int_{t}^{x+h} [f(\xi) - f(t)]\, d\xi - \frac{1}{h^{l-1/p}} \int_{x}^{x+h} \frac{dt}{(t-x)^2} \int_{x}^{t} [f(t) - f(\xi)]\, d\xi.$$

Since both terms are estimated in an analogous way, we shall restrict ourselves to estimating one of them:

$$\frac{1}{h^{l-1/p}}\int_x^{x+h}\frac{dt}{(t-x)^2}\int_x^t\frac{f(t)-f(\xi)}{(t-\xi)^{l+1/p}}(t-\xi)^{l+1/p}\,d\xi\le$$
$$\le\frac{C_1}{h^{l-1/p}}\int_x^{x+h}\frac{dt}{(t-x)^{1-l}}\left|\int_x^t\frac{|f(t)-f(\xi)|^p}{|t-\xi|^{1+pl}}\,d\xi\right|^{1/p}\le$$
$$\le\frac{C_1}{h^{l-1/p}}\left|\int_x^{x+h}\frac{dt}{|t-x|^{(1-l)p'}}\right|^{1/p'}\left|\int_x^{x+h}dt\int_x^t\frac{|f(t)-f(\xi)|^p}{|t-\xi|^{1+pl}}\,d\xi\right|^{1/p}=$$
$$=C_2\left|\int_x^{x+h}dt\int_x^t\frac{|f(t)-f(\xi)|^p}{|t-\xi|^{1+pl}}\,d\xi\right|^{1/p}\underset{h\to0}{\longrightarrow}0,$$

which was to be proved.

In conclusion I express my gratitude to V. P. Il' in, who gave many useful suggestions.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
of the Academy of Sciences of the USSR

Received
18 III 1960

REFERENCES CITED

[1] A. Aronszajn, *Studies in Eigenvalue Problems*, No. 14, Univ. of Kansas, 1955.
[2] E. Gagliardo, Rend. Sem. Mat. di Padova, **27** (1957).
[3] L. N. Slobodetskii, Scientific Notes of the Leningrad Pedagogical Institute, **197**, 54 (1958).
[4] L. N. Slobodetskii, DAN, **120**, No. 3, 468 (1958).
[5] L. N. Slobodetskii, DAN, **123**, No. 4, 616 (1958).
[6] O. V. Besov, DAN, **126**, No. 6, 1163 (1959).
[7] S. V. Uspenskii, DAN, **130**, No. 5, 992 (1960).
[8] V. P. Il' in, Tr. Mat. inst. im. V. A. Steklova AN SSSR, **53**, 128 (1959).
[9] J. Marcinkiewicz, Ann. Soc. Polon. Math., **17**, 42 (1938).

Note: Figure translations are in progress. See original paper for figures.