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Abstract

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ON THE DIMENSION OF PREIMAGES UNDER MAPPINGS OF COMPACTA INTO POLYHEDRA

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Let f be a continuous mapping of a compactum X into a Euclidean space Y, $\dim Y \leq \dim X$. Then [1], by an arbitrarily small displacement of the mapping f, one can ensure that the complete preimage of any point $y \in Y$ has dimension $\leq \dim X - \dim Y$. If, however, Y is an arbitrary polyhedron, the matter becomes more complicated. Suppose, for example, that Y consists of two n-dimensional simplexes with one common vertex y, and X is a sphere ($\dim X \geq n$). Then, if $f : X \to Y$ is such a continuous mapping that the set $f(X)$ contains interior points of both simplexes, the preimage of the point y has dimension $\geq \dim X - 1$ (since it separates the sphere), and by no small "shaking" of the mapping f can this dimension be lowered. Thus, in the given case, to the number $\dim X - \dim Y$ one adds the number $\dim X - 1$. In general, to the difference $\dim X - \dim Y$ one adds a certain number $\chi_0(y)$, depending on the structure of the polyhedron Y near the point y (see the formulation of the main theorem).

Let P be some polyhedron and let χ be a nonnegative integer-valued function defined on it with values $\leq \dim P$. Denote by M_i, $i = 0, 1, 2, \dots, \dim P$, the set of all points $x \in P$ for which $\chi(x) \geq i$. We shall call the function $\chi(x)$ **defective** if: 1) all M_i are closed and 2) for any mapping $\varphi : Q \to P$, where Q is a polyhedron of dimension $\geq \dim P$, and for any $\eta > 0$ there exists a mapping $\psi : Q \to P$ such that $\rho(\varphi, \psi) < \eta$ and

$$\dim(\psi^{-1}(x)) \leq \dim Q - \dim P + \chi(x)$$

for any point $x \in N_i(\eta) = M_i \setminus U(M_{i+1}, \eta)$, where $U(M_{i+1}, \eta)$ is the η-neighborhood of the set M_{i+1} in the polyhedron P.

Theorem 1. *Let $\chi(x)$ be a defective function defined on a polyhedron P; let X be some compactum of dimension $n \geq \dim P$. Then, whatever the mapping $f : X \to P$ and the number $\varepsilon > 0$ may be, there exists a mapping $g : X \to P$ such that $\rho(f, g) < \varepsilon$ and*

$$\dim(g^{-1}(x)) \leq \dim X - \dim P + \chi(x)$$

for every point $x \in P$.

Proof. Construct a sequence of n-dimensional simplicial partitions $\{K_i\}$ in E^{2n+1}, which are nerves of increasingly fine open coverings of the compactum X, and a sequence of continuous mappings $\psi_i : V_i \to P$, where $\{V_i\}$ is a decreasing sequence of open sets of the space E^{2n+1}, with $V_i \supset |K_i|$. The barycentric mappings (see [(2)], p. 208) $X \to |K_i|$ will be denoted by φ_i, $i = 1, 2, \ldots$. We choose the elements K_1, V_1, ψ_1 arbitrarily, subjecting them to the sole condition: $\rho(f_1, \psi_1 \circ \varphi_1) < \varepsilon/2$. Suppose that $K_1, \ldots, K_\nu; V_1, \ldots, V_\nu; \psi_1, \ldots, \psi_\nu$ have already been constructed ($\nu \geq 1$). Suppose, moreover, that certain positive numbers $\omega_1, \omega_2, \ldots, \omega_{\nu-1}$ have been defined inductively. Put

$$\eta_\nu = \min\left(\frac{\varepsilon}{2^{\nu+2}}, \frac{\omega_1}{2^{\nu+1}}, \frac{\omega_2}{2^\nu}, \ldots, \frac{\omega_{\nu-1}}{2^3}\right)$$

and choose so small a positive-

some γ_ν such that $\rho(\psi_\nu(y), \psi_\nu(y')) < \eta_\nu$ when $\rho(y, y') < \gamma_\nu$ ($y, y' \in \overline{V}_\nu$). Now choose a sufficiently fine open covering of the compactum X so that its nerve, which we take for $K_{\nu+1}$, can be realized in $V_\nu \cap (|K_\nu|, \eta_\nu)$, and so that at the same time $\rho(\varphi_\nu, \varphi_{\nu+1}) < \gamma_\nu$ (for the construction of such nerves see the proof of Theorem 3 in ([3])). Since $\chi(x)$ is a defect function, there exists a mapping $\xi_{\nu+1} : |K_{\nu+1}| \to P$ satisfying the condition $\rho(\psi_\nu(y'), \xi_{\nu+1}(y')) < \eta_\nu$, $y' \in |K_{\nu+1}|$, and having the property that

$$\dim(\xi_{\nu+1}^{-1}(x)) \leq \dim X - \dim P + \chi(x)$$

for every point $x \in N_j(\eta_\nu)$, $j = 0, 1, 2, \ldots, \dim P$.

Extend $\xi_{\nu+1}$ to a mapping $\psi_{\nu+1} : \overline{W}_{\nu+1} \to P$ of the closure of some neighborhood $W_{\nu+1} \subset E^{2n+1}$ of the polyhedron $|K_{\nu+1}|$. Then there exists a neighborhood $W'_{\nu+1} \subset W_{\nu+1}$ of the polyhedron $|K_{\nu+1}|$ and a number $\omega_\nu > 0$ such that the complete preimage of the neighborhood $U(x, \omega_\nu)$ of any point $x \in N_j(\eta_\nu)$ under the mapping $\psi_{\nu+1} : W'_{\nu+1} \to P$ admits a $1/2^\nu$-covering of multiplicity $\leq \dim X - \dim P + \chi(x) + 1$. There exists, further, a positive number $\delta_\nu < \gamma_\nu$ such that, when $\rho(y, y') < \delta_\nu$, where $y, y' \in W'_{\nu+1}$, we shall have $\rho(\psi_{\nu+1}(y), \psi_{\nu+1}(y')) < \eta_\nu$. Now take for $V_{\nu+1}$ the neighborhood $U(|K_{\nu+1}|, \delta_\nu) \cap W'_{\nu+1} \cap V_\nu$ of the polyhedron $|K_{\nu+1}|$, and consider the mapping $\psi_{\nu+1}$ only on this set.

Thus the sequences $\{K_i\}$, $\{V_i\}$, $\{\psi_i\}$, $\{\omega_i\}$ have been constructed inductively. In doing so we may assume that the conditions (a)–(i) used in the proof of Theorem 3 in ([3]) are satisfied. Then the polyhedra $|K_i|$ converge to some compactum X', and the mappings φ_i to a homeomorphic mapping φ of the compactum X onto X'.

It is not difficult to see, further, that for any point $x \in X'$ and for any $i < \nu$ the inequality

$$\rho(\psi_{\nu+1}(x), \psi_\nu(x)) < \omega_i/2^{\nu+1}$$

holds; from this it follows that the mappings $\psi_i : X' \to P$ converge to some continuous mapping $\psi : X' \to P$ satisfying the condition

$$\rho(\psi_{\nu+1}(x), \psi(x)) < \omega_\nu \quad (x \in X'). \tag{*}$$

We now show that the mapping $g = \psi \circ \varphi : X \to P$ is the required one. We have: $\lim(\psi_i \circ \varphi_i) = g$. Further, for $x \in X$,

$$\rho(\psi_{\nu+1}(\varphi_{\nu+1}(x)), \psi_\nu(\varphi_\nu(x))) \leq \rho(\psi_{\nu+1}(\varphi_{\nu+1}(x)), \psi_\nu(\varphi_{\nu+1}(x))) + \rho(\psi_\nu(\varphi_{\nu+1}(x)), \psi_\nu(\varphi_\nu(x))) \leq \eta_\nu + \eta_\nu = \varepsilon/2^{\nu+1}.$$

Thus,

$$\rho(\psi_{\nu+1} \circ \varphi_{\nu+1}, \psi_\nu \circ \varphi_\nu) \leq \frac{\varepsilon}{2^{\nu+1}}, \quad \nu = 1, 2, \dots; \qquad \rho(\psi_1 \circ \varphi_1, f) < \frac{\varepsilon}{2},$$

whence, by passage to the limit, we obtain $\rho(f, g) < \varepsilon$.

Let $x \in P$ be an arbitrary point. Since $\rho(x, M_{\chi(x)+1}) > 0$, and the numbers η_ν decrease without bound, for all sufficiently large ν we have $x \in N_{\chi(x)}(\eta_\nu)$. Therefore the complete preimage of the neighborhood $U(x, \omega_\nu)$ under the mapping $\psi_{\nu+1} : V_{\nu+1} \to P$ admits a $\frac{1}{2^\nu}$-covering of multiplicity $\leq \dim X - \dim P + \chi(x) + 1$. The set $\psi^{-1}(x)$, contained (see (*)) in the indicated complete preimage, admits the same covering. Hence $\dim \psi^{-1}(x) \leq \dim X - \dim P + \chi(x)$. It remains to note that the sets $g^{-1}(x)$ and $\psi^{-1}(x)$ are homeomorphic, for $g^{-1}(x) = \varphi^{-1}(\psi^{-1}(x))$, and φ is a homeomorphism. Thus Theorem 1 is proved.

We proceed to the construction of a certain defect function connected with the homotopy properties of the polyhedron P. Let P be a polyhedron, piecewise linearly situated in a Euclidean space E, and let x be an arbitrary point of it. Consider in E the sphere Σ_x with center at the point x and of so small a radius that the sphere Σ_x intersects only those simplices of P whose closures contain the point x. The intersection

$S_x(P) = P \cap \Sigma_x$ will be called the spherical representative of the point x in P. The polyhedron $S_x(P)$ is isomorphic to the boundary of the open star of the point x. By $r(x)$ we shall denote the maximal one of those numbers r such that, for all points $y \in P$ sufficiently close to x, the polyhedron $S_y(P)$ is aspherical in dimensions $< r$ (by asphericity in dimension 0 we mean connectedness).

Lemma. *Let P and Q be two simplicial decompositions, of dimensions s and n, respectively, and let the mesh of the decomposition P be less than δ. Denote by $M_0, M_1, \dots, M_{s-1}$ the subsets of the polyhedron P constructed for the function $\chi_0(x)$ as indicated above. Put, further, $\Sigma_i^0 = M_i$ $(i = 0, 1, \dots, s-1)$, and denote by Σ_i^ν the union of all closed simplices of the decomposition P that do not meet $\Sigma_i^{\nu-1}$ $(\nu = 1, 2, \dots)$. Finally, let $f : Q \to P$ be an arbitrary continuous mapping; let i, k be integers satisfying the conditions $0 \leq k \leq s$, $0 \leq i \leq k+1$. Then there exist a simplicial subdivision $Q_k^{(i)}$ of the decomposition Q and a simplicial mapping $f_k^{(i)} : Q_k^{(i)} \to P$ such that the following conditions are satisfied:*

1) $\rho(f, f_k^{(i)}) < 2(ks + s + i)\delta$;

2) $\dim f_k^{(i)}(T) \geq \dim T - n + j$, if T is such an (open) simplex of the decomposition $Q_k^{(i)}$ that

$$f_k^{(i)}(T) \subset M_{s-j} \setminus \Sigma_{s-j-i}^{ks+s+i} \quad (j = 1, 2, \dots, k);$$

3) if T is such an (open) simplex of the decomposition $Q_k^{(i)}$ that

$$f_k^{(i)}(T) \subset P \setminus \Sigma_{s-k}^{ks+s+i},$$

then

$$\dim f_k^{(i)}(T) \geq \begin{cases} \dim T - n + k + 1 & \text{if } \dim T < n - k + i - 1, \\ \dim T - n + k & \text{otherwise.} \end{cases}$$

If, moreover, the mapping f itself is simplicial, then one may assume that $f_k^{(i)} = f$ on all simplices satisfying conditions 2) and 3) (i.e. in passing from the decomposition Q to $Q_k^{(i)}$ such simplices are not subdivided and on them the mapping $f_k^{(i)}$ coincides with f).

We shall carry out the construction of the mappings $f_k^{(i)}$ successively, increasing the indices i and k. For $i = k = 0$ conditions 2) and 3) impose no requirements, and therefore for $f_0^{(0)}$ one may take the corresponding simplicial approximation to the mapping f (or the identity mapping, if it is simplicial). Further, if the mapping $f_k^{(k+1)}$ has already been constructed, then we may put $f_{k+1}^{(0)} = f_k^{(k+1)}$, since in this case all the conditions, as is easy to see, are satisfied. It remains to carry out the construction of the mapping $f_k^{(i+1)}$, assuming that the mapping $f_k^{(i)}$ has been constructed ($0 \leq i \leq k$).

Let T be an open simplex of dimension $n - k + i$, satisfying the condition

$$f_k^{(i)}(T) \subset P \setminus \Sigma_{s-k}^{ks+s+i+1}.$$

For every $(n - k + i - 1)$-dimensional face of the simplex T condition 3) is fulfilled, and therefore $\dim f_k^{(i)}(\dot{T}) \geq i$. If $\dim f_k^{(i)}(T) \geq i + 1$, then we may assume that the simplex T is not subdivided in passing from $Q_k^{(i)}$ to $Q_k^{(i+1)}$ and that $f_k^{(i)} = f_k^{(i+1)}$ on T. Consider the case $\dim f_k^{(i)}(T) = i$ and denote the simplex $f_k^{(i)}(T)$ by τ. Denote by $\mathfrak{Z}(T)$ and $\mathfrak{Z}(\tau)$ the closures of the open stars of the simplices T and τ, respectively, in the decompositions $Q_k^{(i)}$ and P, by $\dot{\mathfrak{Z}}(T)$ and $\dot{\mathfrak{Z}}(\tau)$ the boundaries of these open stars, and by $\Pi(T)$ and $\Pi(\tau)$ their representatives (i.e. $\Pi(T) * T = \mathfrak{Z}(T)$, $\Pi(\tau) * \tau = \mathfrak{Z}(\tau)$, where $*$ is the sign of the combinatorial join). It is easy to see that $f_k^{(i)}(\Pi(T)) \subset \Pi(\tau)$ and that the mapping $f_k^{(i)} : \Pi(T) \to \Pi(\tau)$ is homeomorphic (see condition 3) in the formulation of the lemma).

Let b be an arbitrary interior point of the simplex T. Then the pyramid $b * \Pi(T)$ is a simplicial decomposition of dimension $\leq k - i$. Since the polyhedron $\Pi(\tau)$ is aspherical in dimensions $< k - i$ (this follows easily from the relation $\tau \subset M_{s-k-1} \setminus M_{s-k} \subset P \setminus \Sigma_{s-k}^{ks+s+i+1}$ on the basis of Hurewicz's theorem and theorem (3), p. 51 of paper (4)), the mapping

$f_k^{(i)} : \Pi(T) \to \Pi(\tau)$ can be extended to a continuous mapping $\varphi : b * \Pi(T) \to \Pi(\tau)$. We may assume that φ is a **simplicial** mapping of some simplicial subdivision B of the polyhedron $b * \Pi(T)$ into the carrier $\Pi(\tau)$, and all simplices of the carrier $\Pi(T)$ enter the subdivision B unsubdivided. Moreover, we may assume that the simplicial mapping $\varphi : B \to \Pi(T)$ is a **homeomorphism**; this follows from the lemma being proved (as applied to the mapping $\varphi : B \to \Pi(\tau)$) for those values of i, k for which, by induction, we assume the lemma proved.

Let us now denote the boundary of the simplex T by $\dot{T}$. Then the combinatorial join $\dot{T} * B$ is a subdivision of the star $\mathfrak{Z}(T)$, and the boundary $\dot{\mathfrak{Z}}(T)$ of this star is not subdivided. Define the mapping $f_k^{(i+1)}$ on the star $\mathfrak{Z}(T)$, taking it on $\dot{T}$ to coincide with $f_k^{(i)}$, and on B to coincide with φ, and extending it to the simplices of the subdivision $T * B$ by linearity. It is then easy to see that condition 3) for the mapping $f_k^{(i+1)}$, considered on $\mathfrak{Z}(T)$, is fulfilled, and moreover $f_k^{(i)} = f_k^{(i+1)}$ on $\dot{\mathfrak{Z}}(T)$. Carrying out this construction for all $(n-k+i)$-dimensional simplices T for which $\dim(f_k^{(i)}(T)) \subset P \setminus \Sigma_{s-k}^{ks+s-i+1}$, we put $f_k^{(i)} = f_k^{(i+1)}$ on all remaining simplices. The constructed mapping $f_k^{(i+1)}$ satisfies conditions 2), 3). Condition 1) follows from the fact that $\rho(f_k^{(i)}, f_k^{(i+1)}) < 2\delta$, for both the mapping $f_k^{(i)}$ and the mapping $f_k^{(i+1)}$ carry the star $\mathfrak{Z}(T)$ into the star $\mathfrak{Z}(\tau)$. The induction just carried out proves the lemma. The final assertion of the lemma is easily verified.

Theorem 2. The function $\chi_0(x) = \dim P - r(x) - 1$ is deficient.
This theorem follows immediately from the lemma proved, for $k = s - 1,\ i = s$, if δ is chosen so small that $\rho(f, f_{s-1}^s) < \eta$ and $\Sigma_i^{s^2+s} \subset U(M_i, \eta)$.

Comparing Theorems 1 and 2, we obtain the following result:

Main theorem. The function $\chi_0(x) = \dim P - r(x) - 1$ has the following property. Whatever finite-dimensional compactum X, continuous mapping $f : X \to P$, and positive number ε may be, there exists a mapping $g : X \to P$ such that $\rho(f, g) < \varepsilon$ and

$$\dim(g^{-1}(x)) \leq \dim X - \dim P + \chi_0(x) = \dim X - r(x) - 1$$

for every point $x \in P$.

In conclusion, we note that for a number of simple polyhedra P the function $\chi_0(x)$ is the least of the functions $\chi(x)$ for which the assertion of the main theorem is valid (cf. the example given at the beginning of the note). Whether the function χ_0 is the least in all cases is unknown to the author.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.