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Abstract

Full Text

HYDROMECHANICS

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ON THE APPLICATION OF THE METHOD OF SINGULARITIES TO THE CALCULATION OF FLUID FLOWS IN RADIAL-AXIAL TURBINES

(Presented by Academician V. I. Smirnov, 30 VI 1958)

The method of singularities has found broad application in solving the inverse and direct problems for axial turbines (¹⁻³); the possibilities of its application to the calculation of flows in radial turbines are also evident. However, only recently K. A. Kiselev (⁴), relying on the work (⁵), used this method as applied to the inverse problem for radial-axial turbines. K. A. Kiselev did not succeed in constructing solutions with singularities in a sufficiently broad class of cases.

In the present note we indicate the apparatus necessary for applying the method of singularities to the calculation of flows in radial-axial turbines of general form.

1. We shall assume the work (⁵) to be known. Denote by $h(\xi)$ the function $\delta(\xi)/\delta(0)$, introduced in (⁵), and we shall use the notation and coordinates introduced in that work. In (⁵) it is shown that in those regions where there are no sources and vortices the equations

$$\begin{aligned} \frac{\partial}{\partial \xi} [h(\xi)R(\xi)v_{\xi}] + \frac{\partial}{\partial \vartheta} [h(\xi)R(\xi)v_{\vartheta}] &= 0, \\ \frac{\partial}{\partial \xi} [R(\xi)v_{\vartheta}] - \frac{\partial}{\partial \vartheta} [R(\xi)v_{\xi}] &= 0, \end{aligned} \tag{1}$$

are valid, from which the velocity components v_{ξ} and v_{ϑ} are to be found.

If we are interested in the flow caused in the plane by a lattice of sources and vortices located at the points $\xi = \xi_0$, $\vartheta = \vartheta_0 + kt$ ($k = 0, \pm 1, \pm 2, \dots$; $t = 2\pi/n$; n is the number of blades), and if the sources and vortices have intensity Q and circulation Γ , then equations (1) should be replaced by the equations

$$\frac{\partial}{\partial \xi} [h(\xi)R(\xi)v_{\xi}] + \frac{\partial}{\partial \vartheta} [h(\xi)R(\xi)v_{\vartheta}] = Q\delta(\xi - \xi_0)\delta_t(\vartheta - \vartheta_0),$$

$$\frac{\partial}{\partial \xi} [R(\xi)v_\vartheta] - \frac{\partial}{\partial \vartheta} [R(\xi)v_\xi] = \Gamma \delta(\xi - \xi_0) \delta_t(\vartheta - \vartheta_0), \quad (2)$$

where $\delta(u)$ is the Dirac delta function, and $\delta_t(u)$ is the periodic Dirac delta function with period t .

2. The solution of equations (2) corresponding to a lattice of sources and vortices will be the sum of the solutions corresponding to the lattice of sources and to the lattice of vortices.

In the case of a lattice of sources ($\Gamma = 0$) there exists a velocity potential Φ such that

$$R(\xi)v_\xi = \frac{\partial \Phi}{\partial \xi}, \quad R(\xi)v_\vartheta = \frac{\partial \Phi}{\partial \vartheta}, \quad (3)$$

and for it we have the equation

$$\frac{\partial}{\partial \xi} \left[h(\xi) \frac{\partial \Phi}{\partial \xi} \right] + \frac{\partial}{\partial \vartheta} \left[h(\xi) \frac{\partial \Phi}{\partial \vartheta} \right] = Q \delta(\xi - \xi_0) \delta_t(\vartheta - \vartheta_0). \quad (4)$$

In the case of a vortex lattice ($Q = 0$) there exists a stream function Ψ such that

$$h(\xi)R(\xi)v_\xi = -\frac{\partial \Psi}{\partial \vartheta}, \quad h(\xi)R(\xi)v_\vartheta = +\frac{\partial \Psi}{\partial \xi}, \quad (5)$$

and for it we have the equation

$$\frac{\partial}{\partial \xi} \left[\frac{1}{h(\xi)} \frac{\partial \Psi}{\partial \xi} \right] + \frac{\partial}{\partial \vartheta} \left[\frac{1}{h(\xi)} \frac{\partial \Psi}{\partial \vartheta} \right] = \Gamma \delta(\xi - \xi_0) \delta_t(\vartheta - \vartheta_0). \quad (6)$$

The boundary conditions for $\xi = \pm\infty$ have the form

$$\begin{aligned} h(\xi) \frac{\partial \Phi}{\partial \xi} \Big|_{\xi=-\infty} &= -\frac{Q}{2t}, & \frac{\partial \Phi}{\partial \vartheta} \Big|_{\xi=-\infty} &= 0, \\ h(\xi) \frac{\partial \Phi}{\partial \xi} \Big|_{\xi=+\infty} &= +\frac{Q}{2t}, & \frac{\partial \Phi}{\partial \vartheta} \Big|_{\xi=+\infty} &= 0; \\ \frac{1}{h(\xi)} \frac{\partial \Psi}{\partial \xi} \Big|_{\xi=-\infty} &= -\frac{\Gamma}{2t}, & \frac{\partial \Psi}{\partial \vartheta} \Big|_{\xi=-\infty} &= 0, \end{aligned} \quad (7)$$

$$\frac{1}{h(\xi)} \frac{\partial \Psi}{\partial \xi} \Big|_{\xi=+\infty} = +\frac{\Gamma}{2t}, \quad \frac{\partial \Psi}{\partial \vartheta} \Big|_{\xi=+\infty} = 0. \quad (8)$$

From (4), (6), (7), and (8) we see that the problems of determining Φ and Ψ differ only by replacing $h(\xi)$ by $\frac{1}{h(\xi)}$ and Q by Γ . Therefore we shall consider only the problem of determining Φ .

3. The solution of the problem under consideration with the periodic Dirac function δ_t on the right-hand side of equation (4) can be obtained by replacing δ_t on the right-hand side of (4) by the function $\delta_t^{(m)}$, which is the kernel of the Dirichlet integral, solving the problem for equation (4) with the changed right-hand side, and letting m tend to infinity.

Since

$$\delta_t^{(m)}(\vartheta - \vartheta_0) = \frac{2}{t} \left\{ \frac{1}{2} + \sum_{k=1}^m \cos \frac{2k\pi}{t} (\vartheta - \vartheta_0) \right\}, \quad (9)$$

the solution $\Phi^{(m)}$ of the problem with the changed right-hand side can be sought in the form

$$\Phi^{(m)} = \frac{A_0(\xi)}{2} + \sum_{k=1}^m A_k(\xi) \cos \frac{2k\pi}{t} (\vartheta - \vartheta_0). \quad (10)$$

For $A_k(\xi)$ we have the equation

$$\frac{d}{d\xi} \left[h(\xi) \frac{dA_k}{d\xi} \right] - \left(\frac{2k\pi}{t} \right)^2 h(\xi) A_k = \frac{2Q}{t} \delta(\xi - \xi_0). \quad (11)$$

The conditions (7), written for $\Phi^{(m)}$, first of all mean that $\partial \Phi^{(m)} / \partial \xi$ and $\partial \Phi^{(m)} / \partial \vartheta$ for $\xi = \pm\infty$ do not depend on ϑ .

This gives the conditions

$$A_k(\xi) \Big|_{\xi=\pm\infty} = 0, \quad \frac{dA_k}{d\xi} \Big|_{\xi=\pm\infty} = 0 \quad (k = 1, 2, \dots). \quad (12)$$

When the conditions (12) are satisfied, the conditions (7) for $\Phi^{(m)}$ will be satisfied if

$$h(\xi) \frac{dA_0}{d\xi} \Big|_{\xi=-\infty} = -\frac{Q}{t}, \quad h(\xi) \frac{dA_0}{d\xi} \Big|_{\xi=+\infty} = +\frac{Q}{t}. \quad (13)$$

From (11), (12), and (13) we see that the coefficients $A_k(\xi)$ are functions of the Green's-function type, and that the entire difficulty of finding them consists in finding linearly independent solutions of the homogeneous equations (11).

The required function Φ is obtained from $\Phi^{(m)}$ when m is infinitely large. For Φ we have

$$\Phi = \frac{A_0(\xi)}{2} + \sum_{k=1}^{\infty} A_k(\xi) \cos \frac{2k\pi}{t} (\vartheta - \vartheta_0). \quad (14)$$

4. For the practical integration of the homogeneous equations (11), it is useful to make use of a certain qualitative characteristic of flows through cascades. Namely, it is known from theoretical considerations and experiment that, in flow through a cascade, at a distance of $1-1\frac{1}{2}$ pitches in front of the cascade and behind the cascade, the velocity components are practically independent of the transverse coordinate ϑ . Since, in practice, knowledge of the flow is of interest only for those values of ξ where the flow still depends on the coordinate ϑ , this circumstance makes it possible to simplify the integration of equations (11) by an appropriate change of the function $h(\xi)$.

Let the independence of the velocity components from the coordinate ϑ hold with sufficient accuracy for $\xi \leq a$ and $\xi \geq b$. We shall use the actual values of the function $h(\xi)$ in the interval (a, b) , and outside this interval we shall pass the function $h(\xi)$ smoothly into a constant (different from zero), and shall assume that precisely this modified function is substituted for $h(\xi)$ in equations (11).

For $k = 0$, the homogeneous equation (11) has the following linearly independent solutions:

$$A_0^{(1)}(\xi) = \int \frac{d\xi}{h(\xi)}, \quad A_0^{(2)}(\xi) = 1. \quad (15)$$

Consequently, the function $A_0(\xi)$ is easily constructed up to an arbitrary constant term.

For $k \neq 0$, put

$$A_k(\xi) = \sqrt{h(\xi)} B_k(\xi). \quad (16)$$

Then, from the homogeneous equation for A_k , we obtain the homogeneous equation for B_k

$$\frac{d^2 B_k}{d\xi^2} - [(kn)^2 + H(\xi)] B_k = 0, \quad (17)$$

where $H(\xi)$ is different from zero only in a finite interval (for the modified $h(\xi)$) and is given by the formula

$$H(\xi) = \frac{1}{2} \left[\frac{h'(\xi)}{h(\xi)} \right]' + \frac{1}{4} \left[\frac{h'(\xi)}{h(\xi)} \right]^2. \quad (18)$$

Estimates show that the function $H(\xi)$, in cases of practical interest, proves to be a quantity of order unity. At the same time, the number of blades is $n = 10-20$. Therefore, even for small k , the first term in the square brackets of equation (17) is large in comparison with the second. This makes it possible to use approximate methods for integrating equation (17) and to write explicit expressions for the linearly independent solutions.

As approximate linearly independent solutions one may, for example, use the functions

$$\begin{aligned} B_k^{(1)}(\xi) &= e^{-kn(\xi-\xi_0)} \left[1 - \frac{a_1(\xi, \xi_0)}{(kn)} + \frac{a_2(\xi, \xi_0)}{(kn)^2} \right], \\ B_k^{(2)}(\xi) &= e^{kn(\xi-\xi_0)} \left[1 + \frac{a_1(\xi, \xi_0)}{(kn)} + \frac{a_2(\xi, \xi_0)}{(kn)^2} \right], \end{aligned} \quad (19)$$

where

$$\begin{aligned} a_1(\xi, \xi_0) &= \frac{1}{2} \int_{\xi_0}^{\xi} H(\eta) d\eta, \\ a_2(\xi, \xi_0) &= \frac{1}{4} \int_{\xi_0}^{\xi} \left[H(\eta) \int_{\xi_0}^{\eta} H(\eta) d\eta \right] d\eta - \frac{H(\xi) - H(\xi_0)}{4}. \end{aligned} \quad (20)$$

Having the solutions (19), we can without difficulty write out the functions $B_k(\xi)$:

$$B_k(\xi) = \begin{cases} -\frac{nQ}{2\pi} \sqrt{\frac{1}{h(\xi_0)}} \frac{1}{kn} B_k^{(1)}(\xi), & \xi > \xi_0, \\ -\frac{nQ}{2\pi} \sqrt{\frac{1}{h(\xi_0)}} \frac{1}{kn} B_k^{(2)}(\xi), & \xi < \xi_0. \end{cases} \quad (21)$$

Having $B_k(\xi)$, we find $A_k(\xi)$ and write out the solution Φ . In writing out the solution it is convenient to use the functions

$$S_m(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1}{k^m} e^{-k\alpha} \cos k\beta, \quad \sigma_m(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1}{k^m} e^{-k\alpha} \sin k\beta, \quad (22)$$

through which Φ , $\partial\Phi/\partial\xi$, and $\partial\Phi/\partial\vartheta$ are simply expressed. The functions S_0 and σ_0 are expressed in an obvious way through elementary functions. Functions with larger indices are obtained from functions with smaller indices by integration.

6. Having the flows from a lattice of sources and a lattice of vortices, as well as a flow not depending on ϑ , we have the entire apparatus necessary for using the method of singularities.

I consider it my pleasant duty to express gratitude to Academician V. I. Smirnov for useful discussion of the work.

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