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Abstract

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A CRITERION FOR A COMPLETELY UNIFORMLY DISTRIBUTED SEQUENCE

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In the present paper a criterion is given for a sequence of real numbers α, taken from the interval $[0,1]$: $\alpha = \alpha_1, \alpha_2, \alpha_3, \ldots$, to be completely uniformly distributed.* Our arguments will be analogous to the arguments of § 4 of paper ([2]).

By $\Delta_s = (\delta_1, \delta_2, \ldots, \delta_s)$ we shall denote a parallelepiped lying in the s-dimensional unit cube, defined by the condition that the i-th coordinate of its points belongs to the interval δ_i; by $|\Delta_s|$ we shall denote the volume of the parallelepiped Δ_s.

Theorem. Let the sequence α be such that there exists a constant c such that, for any $s \geqslant 1$ and any Δ_s,

$$\lim_{P\to\infty} \frac{N_P(\Delta_s)}{P} < c|\Delta_s|.$$

Then the sequence α is completely uniformly distributed.

Lemma 1. Let $r \geqslant 1$ be an integer. Let δ be some interval in $[0,1]$, and let $|\delta|$ be its length. Consider the unit cube in l-dimensional space. The Lebesgue measure of those points of this cube for which the number of coordinates falling in the interval δ (denote this number by A_δ^l) satisfies the inequality

$$\left|A_\delta^l - l|\delta|\right| \geqslant \frac{l}{r},$$

does not exceed $r^4/4l^2$.

Proof. There are C_l^k ways by which one can distribute k marks "fell in δ" among l positions. Let δ be the interval $[a,b]$. The volume of the region defined by the condition that k specified coordinates satisfy the inequalities $a \leqslant \xi \leqslant b$, and the remaining $l-k$ the inequalities either $0 \leqslant \xi \leqslant a$, or $b < \xi \leqslant 1$, is equal to $(b-a)^k(1-(b-a))^{l-k} = |\delta|^k(1-|\delta|)^{l-k}$. Therefore the volume of the region of points k of whose coordinates have fallen in δ, and the remaining ones outside δ, is $C_l^k|\delta|^k(1-|\delta|)^{l-k}$. Hence the measure sought in the lemma is equal to

$$L = \sum_{|k-l|\delta|| \geqslant l/r} C_l^k |\delta|^k (1-|\delta|)^{l-k}.$$

Carrying out the usual estimate of this expression, we obtain the lemma.

* For the definition of a completely uniformly distributed sequence, see paper ([1]).

Let $s \geqslant 1$ and $l \geqslant 1$ be natural numbers. Consider the unit cube in ls-dimensional space. Let $\Delta_s = (\delta_1, \dots, \delta_s)$ be a fixed parallelepiped in the unit cube of s-dimensional space; $|\Delta_s| = |\delta_1| \cdots |\delta_s|$ is the volume of this parallelepiped. Let some point of the cube have coordinates

$$(a_1, a_2, \dots, a_s, a_{s+1}, \dots, a_{2s}, \dots, a_{(l-1)s+1}, \dots, a_{ls}).$$

Group the coordinates s at a time, $(B_1 \dots B_l)$, where

$$B_k = (a_{(k-1)s+1} \dots a_{ks}), \qquad k = 1, 2, \dots, l,$$

points of the unit cube of s-dimensional space. Denote by $A_{\Delta_s}^{(l)}$ the number of these points that fall into Δ_s.

Lemma 2. Let $r \geqslant 1$ be an integer. The Lebesgue measure of the points of the unit cube of ls-dimensional space for which

$$\left|A_{\Delta_s}^{(l)} - l|\Delta_s|\right| \geqslant \frac{l}{r}$$

does not exceed $r^4/4l^2$.

Proof. We compute the Lebesgue measure of the set of those points of the ls-dimensional cube for which, in the representation $(B_1, \dots, B_l)$, at certain x places there is a hit in Δ_s, and at the remaining places there is not. Let E_s be the unit cube of s-dimensional space. Obviously, this measure is equal to

$$\underbrace{\int_{\Delta_s} \cdots \int_{\Delta_s}}_{x \text{ times}} \underbrace{\int_{E_s/\Delta_s} \cdots \int_{E_s/\Delta_s}}_{l-x \text{ times}} dx_1 \cdots dx_{ls} = |\Delta_s|^x (1-|\Delta_s|)^{l-x}$$

(each integral is s-fold).

Since the x hits can be arranged among the l places in C_l^x ways, the measure sought in the lemma is equal to

$$L = \sum_{\substack{x=0 \\ |x-l|\Delta_s|| \geqslant l/r}}^{l} C_l^x |\Delta_s|^x (1-|\Delta_s|)^{l-x}.$$

Carrying out the usual estimate, we obtain that $L \leqslant r^4/4l^2$.

Proof of the theorem. Let the sequence $\alpha = \alpha_1, \alpha_2, \dots, \alpha_s, \dots$ satisfy the condition of the theorem. Let $\Delta_s = (\delta_1, \dots, \delta_s)$ be a parallelepiped in the unit cube of s-dimensional space. Arrange the numbers of the sequence into groups of s:

$$P_1^{(s)}, P_2^{(s)}, \dots,$$

where

$$P_k^{(s)} = (\alpha_{(k-1)s+1}, \dots, \alpha_{ks}),$$

i.e. consider the sequence of points of the s-dimensional unit cube. From X terms of the sequence we obtain $[X/s]$ points. Denote by $A_{\Delta_s}^{[X/s]}$ the number of times the points $P_j^{(s)}$ $(j = 1, \dots, [X/s])$ fall into Δ_s. Choose arbitrarily $l \geqslant 1$ and arrange the points $P^{(s)}$ into groups of l:

$$\underbrace{P_1^{(s)} P_2^{(s)} \cdots P_l^{(s)}}_{l} \ \underbrace{P_{l+1}^{(s)} P_{l+2}^{(s)} \cdots P_{2l}^{(s)}}_{l} \ \cdots$$

Take a natural $r \geqslant 1$ and call an l-term group of points "good" if the number of points falling into Δ_s is equal to $l(|\Delta_s| + \theta/r)$, $|\theta| \leqslant 1$, and "bad" otherwise. Denote by $M([X/s])$ the number of times bad groups occur in the sequence of groups formed from the first $l\left[\frac{[X/s]}{l}\right]$ numbers of the sequence α.

Then the number of good groups will be

$$\left[\frac{X}{sl}\right] - M\left(\left[\frac{X}{s}\right]\right) = \frac{X}{sl} - M\left(\left[\frac{X}{s}\right]\right) + O(1)$$

with an absolute constant in O. A good group contributes $l(|\Delta_s| + \theta/r)$ points falling into Δ_s. Therefore, among the $[X/s]$ points $P^{(s)}$, the number falling into Δ_s is

$$A_{\Delta_s}^{[X/s]} = l\left(|\Delta_s| + \frac{\theta}{r}\right)\left(\frac{X}{sl} - M\left(\left[\frac{X}{s}\right]\right)\right) + l\theta_1 M\left(\left[\frac{X}{s}\right]\right) + O(l).$$

The term $O(l)$ arises from the fact that, possibly, there is an incomplete group of points. Hence

$$A_{\Delta_s}^{[X/s]} = \frac{X}{s}\left(|\Delta_s| + \frac{\theta}{r}\right) + O\left(M\left(\left[\frac{X}{s}\right]\right) l\right) + O(l).$$

Let $\mathfrak{M}$ denote the set of those points in the ls-dimensional unit cube for which

$$\left|A_{\Delta_s}^{(l)} - l|\Delta_s|\right| \geq l/r.$$

Each bad combination represents a point of $\mathfrak{M}$. Therefore

$$M\left(\left[\frac{X}{s}\right]\right) \leq N_{s[X/s]}(\mathfrak{M}).$$

But for $X \geq X_0$, according to the hypothesis of the theorem,

$$N_{s[X/s]}(\mathfrak{M}) \leq 2C \text{ mes}\, \mathfrak{M}\, X,$$

$$A_{\Delta_s}^{[X/s]} = \frac{X}{s}\left(|\Delta_s| + \frac{\theta}{r}\right) + O\left(X\frac{r^4}{l}\right) + O(l)$$

(by Lemma 2).

Hence, letting X tend to infinity, we obtain

$$\overline{\lim}_{X\to\infty}\left|\frac{A_{\Delta_s}^{[X/s]}}{X/s} - |\Delta_s|\right| \leq \frac{1}{r} + O\left(\frac{sr^4}{l}\right).$$

Now letting the parameter l tend to infinity, we obtain

$$\overline{\lim}_{X\to\infty}\left|\frac{A_{\Delta_s}^{[X/s]}}{X/s} - |\Delta_s|\right| \leq \frac{1}{r}.$$

Consider the s sequences $T^j\alpha,\ j = 0, 1, 2, \dots, s-1$ (among which $T^0\alpha$ is our sequence α),

$$T^j\alpha = \alpha_{j+1}, \alpha_{j+2}, \alpha_{j+3}, \dots$$

Each of these sequences satisfies the condition of the criterion. Denote the quantity $A_{\Delta_s}^{[X/s]}$, constructed for the sequence $T^j\alpha$, by $A_{\Delta_s}^{[X/s]}(T^j\alpha)$. We have

$$\lim_{X\to\infty}\frac{A_{\Delta_s}^{[X/s]}(T^j\alpha)}{X/s} = |\Delta_s|, \qquad j = 0, 1, \dots, s-1.$$

But, obviously,

$$N_X(\Delta_s) = \sum_{j=0}^{s-1} A_{\Delta_s}^{[X/s]}(T^j\alpha) + O(s).$$

Hence

$$\lim_{X\to\infty}\frac{N_X(\Delta_s)}{X} = s\frac{|\Delta_s|}{s} = |\Delta_s|.$$

Since $s \geq 1$ and Δ_s is an arbitrary parallelepiped, the theorem is proved.

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References

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2. A. G. Postnikov, I. I. Pyatetskii, *Izv. AN SSSR, Ser. Mat.*, **21**, No. 4, 501 (1957).

Note: Figure translations are in progress. See original paper for figures.