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Abstract

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ON THE UNIQUENESS OF SOLUTIONS OF DEGENERATING EQUATIONS AND THE RIGIDITY OF SURFACES

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In the theory of infinitesimal bendings one encounters second-order partial differential equations. For example, for surfaces of revolution the tangential component of the displacement vector **U** satisfies the equation [1]

$$\rho v_{tt} - \rho'' v_{\vartheta\vartheta} - \rho'' v = 0, \tag{1}$$

where $\rho = \rho(t)$ is the meridian of the surface. If the surface S is projected one-to-one onto the plane Oxy, then the "vertical component" of the displacement vector satisfies an equation of the form [2]

$$z_{xx}\zeta_{yy} - 2z_{xy}\zeta_{xy} + z_{yy}\zeta_{xx} = 0, \tag{2}$$

where $z = z(x, y)$ is the surface S. Equations (1) and (2) are elliptic, or hyperbolic, or parabolic according as the Gaussian curvature of the surface $K > 0$, or < 0, or $= 0$, respectively.

In the present paper three problems are considered.

1. The Cauchy Problem

Theorem 1. *The solution of the equation*

$$y^m f(y) v_{yy} - v_{xx} + a(x, y) v_x + b(x, y) v_y + c(x, y) v = \varphi(x, y) \tag{3}$$

$$(0 < m < 2)$$

with initial data

$$v(x, 0) = 0, \qquad v_y(x, 0) = 0 \tag{4}$$

exists and is unique in the domain Δ*, bounded by the characteristics* AC*,* BC*, lying in the half-plane* $y > 0$*, and by the segment* AB *of the* x*-axis, if the following conditions are satisfied: 1)* a, b, c *are continuous together with their first derivatives with respect to* x*; 2)* $y^{1-m} b \equiv \delta(x, y)$*,* $\delta(x, y) \to 0$ *as* $y \to 0$*, uniformly with respect to* x*; 3)* $f(y) > 0$*,* $f'(y)$ *is bounded; 4)* $\varphi(x, y) = y^{m/2} \psi(x, y)$*;* $\psi(x, y)$*,* $\psi_x(x, y)$ *are bounded; 5)* v, v_x, v_y *are continuous for* $y \geq 0$*,* $v_y \in \operatorname{Lip} \nu,\ 0 < \nu \leq 1$*.*

In particular, if $\varphi(x, y) \equiv 0$*, then equation (3) under the boundary condition (4) has only the zero solution.*

Proof. The theorem is proved on the basis of the method used by I. S. Berezin [3] and M. H. Protter [4], i.e. by reducing (1) to a system of integral equations.

This result can immediately be transferred to the theory of infinitesimal bendings, since equation (1) is a special case of equation (3).

Theorem 2. Suppose there is a piece of a surface of revolution of negative curvature whose boundary consists of two asymptotic lines AC, BC and a segment of a parabolic line AB, lying in the plane $t = 0$. If the meridian of the surface $\rho(t)$ $(t \geq 0)$ satisfies the conditions: 1) $\rho''(0) = -\infty$, $\rho''(t) = t^{-m} \varphi(t)$ near $t = 0$, where $0 < m < 2$, $\varphi(t) > 0$ and is bounded; 2) $\mathbf{U}|_{AB} = 0$, then this piece of the surface will be rigid.

Definition. A contour of a surface of negative curvature will be called an **admissible contour** if the two asymptotic lines issuing from each point of this contour do not intersect it at another point, except in the case of coincidence with the entire contour.

Theorem 3. Cutting from a torus any piece of a surface of negative curvature, we obtain a surface with contour Γ, lying on the concave part of the torus. If Γ is an admissible contour, then this surface is rigid.

Proof. From the rigidity of the convex part of the torus and the uniqueness of the Cauchy problem and of the Goursat problem for equation (1), where $\rho'' \geq 0$, by the method of gluing we immediately obtain our theorem.

Theorem 4. Suppose there is a piece of a surface of revolution S, on one part S^+ of which the curvature is positive, and on the other part S^- negative. AB is the line of gluing $(t = 0)$, where $K = 0$; L is the boundary of S; $L = \sigma + \Gamma$; σ is the boundary of S^+; Γ is the boundary of S^-. If: 1) $\rho''(+0) = -\infty$, $-\rho / t\rho'' + 2\rho' / \rho'' < At$ for $t \geq 0$; 2) $\rho''(-0) = +\infty$, $\rho''(t) = |t|^{-m} \varphi(t)$ near $t = -0$, where $0 < m < 2$, $\varphi(t) > 0$; 3) $\mathbf{U}_S = 0$ on σ, then S will be rigid.

Proof. From the work of M. V. Keldysh (^{5}) and Theorem 2 our theorem follows immediately.

2. Problem D and Problem E. First of all we shall solve geometric problems.

Let S be a surface of nonnegative curvature, not containing a plane piece, with contours $L_0, L_1, \ldots, L_n$; let a be a certain constant vector. Suppose S is projected one-to-one onto the plane Π, perpendicular to the vector a. Denote by $\Gamma_0, \Gamma_1, \dots, \Gamma_n$ the projections of the contours $L_0, L_1, \dots, L_n$ onto Π, where Γ_0 is a convex curve, and Γ_j $(j = 1, \dots, n)$ are concave curves lying inside the curve Γ_0. The integral equality (^{6}) holds

$$2 \iint_S (\beta\gamma - \alpha^2)(\mathbf{n}a)\, dS = \int_L (\mathbf{V}\, d\mathbf{V}\, a) \quad (L = L_0 + \dots + L_n), \tag{5}$$

where $\beta\gamma - \alpha^2 \leq 0$, when S, not containing a plane piece, has nonnegative Gaussian curvature, and moreover from $\beta\gamma - \alpha^2 = 0$ it follows that $\alpha = \beta = \gamma = 0$, hence the vector of rotation $\mathbf{V} \equiv$ const (^{2}). Therefore, by virtue of $(\mathbf{n}a) \geq 0$ (everywhere on S), the rigidity of the surface is a consequence of the inequality

$$\int_L (\mathbf{V}\, d\mathbf{V}\, a) \geq 0.$$

Theorem 5. *S will be rigid if, on the aggregate of contours*

$$L = L_0 + L_1 + \dots + L_n$$

there are realized sleeve constraints of the form

$$\mathbf{U}a = 0. \tag{6}$$

Proof. On the contours L_i $(i = 0, 1, \dots, n)$ introduce the trihedron $s_i, m_i, b_i = s_i \times m_i$ $(i = 0, 1, \dots, n)$, where m_i, b_i are the principal normal and binormal of the curve L_i; s_i is the tangent to the curve L_i, and we shall regard the direction on the contour as positive if the surface remains on the left. By virtue of the convexity of the curve Γ_0 and the concavity of the curve Γ_j $(j = 1, \dots, n)$, we shall have $(b_i a) \geq 0$ $(i = 0, 1, \dots, n)$.

Differentiating (6) along L, we obtain $(\mathbf{U}'a) = 0$, here $\mathbf{U}' = d\mathbf{U}/ds$. In view of the fact that $\mathbf{U}' = \mathbf{V} \times s_i$ (^{7}), where $\mathbf{V}$ is the vector of rotation of the bending, we have

$$(\mathbf{V}\mathbf{s}_i\mathbf{a}) \equiv (\mathbf{V}\mathbf{g}_i) = 0 \quad \text{on } L_i \quad (i = 0, 1, \dots, n), \tag{7}$$

where $\mathbf{g}_i \equiv [\mathbf{s}_i \times \mathbf{a}]$, and $\mathbf{g}_i' = k_i[\mathbf{m}_i \times \mathbf{a}]$, $k_i (\geq 0)$ is the curvature of the curve L_i. Differentiating (7) along L_i, we obtain

$$(\mathbf{V}'\mathbf{g}_i) = -(\mathbf{V}\mathbf{g}_i') \quad \text{on } L_i \quad (i = 0, 1, \dots, n). \tag{8}$$

Consider the different cases:

1) $k_i = 0$ or $[\mathbf{m}_i \times \mathbf{a}] = 0$ on some segment of the curve L_i. Then $\mathbf{g}_i' = 0$. From (7), (8) we obtain $(\mathbf{V}\mathbf{g}_i) = 0$, $(\mathbf{V}'\mathbf{g}_i) = 0$; consequently, $\mathbf{V} \parallel \mathbf{V}'$ or $[\mathbf{V} \times \mathbf{V}'] \parallel \mathbf{g}_i \perp \mathbf{a}$. Therefore on this segment we have $(\mathbf{V}\mathbf{V}'\mathbf{a}) = 0$.
2) $(\mathbf{b}_i\mathbf{a}) = 0$, i.e. $\mathbf{g}_i \parallel \mathbf{g}_i'$, or $\mathbf{g}_i' = \lambda \mathbf{g}_i$ on some segment of the curve L_i. Then $(\mathbf{V}'\mathbf{g}_i) = -\lambda(\mathbf{V}\mathbf{g}_i) = 0$. Therefore, as in 1), we obtain $(\mathbf{V}\mathbf{V}'\mathbf{a}) = 0$.
3) All remaining cases, i.e. $k_i > 0$, $(\mathbf{b}_i\mathbf{a}) > 0$, $[\mathbf{m}_i \times \mathbf{a}] \neq 0$. Then

$$[\mathbf{g}_i \times \mathbf{g}_i'] = k_i[\mathbf{s}_i \times \mathbf{a}] \times [\mathbf{m}_i \times \mathbf{a}] = k_i(\mathbf{b}_i\mathbf{a})\mathbf{a}.$$

By virtue of (7), (8) we have

$$(\mathbf{V}\mathbf{V}'\mathbf{a}) = \frac{1}{k_i(\mathbf{b}_i\mathbf{a})}[\mathbf{V} \times \mathbf{V}'][\mathbf{g}_i \times \mathbf{g}_i'] = \frac{1}{k_i(\mathbf{b}_i\mathbf{a})}(\mathbf{V}\mathbf{g}_i')^2 \geq 0.$$

For all these cases we have $(\mathbf{V}\mathbf{V}'\mathbf{a}) \geq 0$ on L_i $(i = 0, 1, \dots, n)$. Therefore the right-hand side of equality (5) is ≥ 0. Our theorem follows from this.

Remark 1. N. V. Efimov proved this theorem for a simply connected surface of positive curvature by means of the maximum principle for an equation of elliptic type $^{(2)}$. Using the integral equality (5), K. P. Grotemeyer proved this theorem only for a simply connected surface of positive curvature with a plane contour $^{(6)}$. Our theorem is a generalization of their results.

If the contour L_0 contains segments of straight lines $\sigma_1, \sigma_2, \dots, \sigma_N$, and the normal to the surface along σ_k $(k = 1, \dots, N)$ is perpendicular to the vector $\mathbf{a}$, i.e. $(\mathbf{a}\mathbf{n}) = 0$ along σ_k $(k = 1, \dots, N)$, then the following holds.

Theorem 6. *S will be rigid if*

$$(\mathbf{U}\mathbf{a}) = 0 \quad \text{on } L - \sum_{k=1}^{N} \sigma_k. \tag{9}$$

Proof. This theorem is an immediate consequence of Theorem 5, if under our assumption we prove that the conditions $(\mathbf{U}\mathbf{a}) = 0$ on σ_k $(k = 1, \dots, N)$ automatically follow from (9).

Resolve the vectors $\mathbf{U}, \mathbf{V}$ with respect to the trihedron $\mathbf{s}, \mathbf{m}, \mathbf{b}$ in the form

$$\mathbf{U} = u_s\mathbf{s} + u_m\mathbf{m} + u_b\mathbf{b}, \qquad \mathbf{V} = v_s\mathbf{s} + v_m\mathbf{m} + v_b\mathbf{b}.$$

Taking into account that $\varkappa \equiv k \equiv 0$ on σ_k $(k = 1, \dots, N)$, we have $^{(7)}$

$$\frac{du_s}{ds} = 0, \qquad v_b = \frac{du_m}{ds}, \qquad v_m = -\frac{du_b}{ds}, \qquad \frac{dv_m}{ds}\cos\theta + \frac{dv_b}{ds}\sin\theta = 0 \quad \text{on } \sigma_k \tag{10}$$

$$(k = 1, \dots, N),$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{m}$, and $\mathbf{n} = \mathbf{m}\cos\theta + \mathbf{b}\sin\theta$. Therefore

$$(\mathbf{an}) = (\mathbf{am})\cos\theta + (\mathbf{ab})\sin\theta = 0 \quad \text{on } \sigma_k \quad (k = 1, \dots, N). \tag{11}$$

By virtue of (10), (11), on σ_k $(k = 1, \dots, N)$ we have

$$\frac{d^2}{ds^2}(\mathbf{Ua}) = ((\mathbf{am})\cos\theta + (\mathbf{ab})\sin\theta)\left(\frac{dv_b}{ds}\cos\theta - \frac{dv_m}{ds}\sin\theta\right) +$$

$$+((\mathbf{am})\sin\theta - (\mathbf{ab})\cos\theta)\left(\frac{dv_b}{ds}\sin\theta + \frac{dv_m}{ds}\cos\theta\right) = 0.$$

By virtue of the continuity of the bending vector $\mathbf{U}$ and condition (9), it follows immediately that $(\mathbf{Ua}) = 0$ on σ_k $(k = 1, \dots, N)$. The theorem is proved.

We pass to the application of the results set forth above to the degenerating equation

$$r\zeta_{yy} - 2s\zeta_{xy} + t\zeta_{xx} = 0. \tag{2'}$$

Theorem 7. Let the surface $z = z(x, y)$, $(x, y) \in D$, satisfy all the conditions of Theorems 5 and 6, $z(x, y) \in C^2(D)$, $rt - s^2 \geq 0$ in $\overline{D}$, and suppose that r, s, t vanish simultaneously only on a set of plane measure zero. Then, under the boundary condition

$$\zeta = 0 \quad \text{on } \Gamma - \sum_{k=1}^{N} \gamma_k,$$

where $\Gamma = \Gamma_0 + \Gamma_1 + \dots + \Gamma_n$ are the contours of the domain D, $\gamma_k \subset \Gamma_0$ $(k = 1, \dots, N)$ are the projections of the lines σ_k onto the plane Π, equation $(2')$ has only the zero solution in the class $C^2(D) \cap C(\overline{D})$.

Remark 2. Theorems 5, 6 and equality (2) are valid for thrice differentiable vectors $\mathbf{U}$ and $\mathbf{r}$. But it is not difficult to see that they are also valid for $\mathbf{U}, \mathbf{r} \in C^2$. This is proved by a limiting passage.

Remark 3. Chaplygin' s equation and Tricomi' s equation are special cases of equation (2). Therefore Theorem 7 also holds for them.

3. Consider the equation

$$y^{2m+1}\zeta_{xx} + \zeta_{yy} = 0, \qquad m \geq 0, \tag{12}$$

in the domain $\{0 \leq x \leq l,\ -a \leq y \leq b\}$, where $a, b, l > 0$. We suppose that

$$\zeta\big|_{y=-0} = \zeta\big|_{y=+0}, \qquad \zeta_y\big|_{y=-0} = \zeta_y\big|_{y=+0}. \tag{13}$$

Theorem 8. Under the boundary conditions

$$\zeta\big|_{y=-a} = F(x), \qquad \zeta\big|_{y=b} = \Phi(x), \qquad \zeta(0,y) = \zeta(l,y),$$

where $F(x)$, $\Phi(x)$ are continuous on $[0,l]$ and have bounded variation on it, with $F(0) = F(l)$, $\Phi(0) = \Phi(l)$, equation (12) has a unique solution if

$$I_{-\beta}(\lambda_k)J_\beta(\mu_k) + I_\beta(\lambda_k)J_{-\beta}(\mu_k) \neq 0, \tag{14}$$

where $J_{\pm\beta}, I_{\pm\beta}$ are Bessel functions of the first and second kinds, $\beta = \dfrac{1}{2m+3}$,

$$\lambda_k = \frac{2k\pi}{l}\frac{2}{2m+3}\,a^{\frac{2m+3}{2}}, \qquad \mu_k = \frac{2k\pi}{l}\frac{2}{2m+3}\,b^{\frac{2m+3}{2}} \quad (k = 1, 2, \ldots).$$

Corollary. Let S have the form

$$z(x,y) = (m+1)(2m+3)x^2 + y^{2m+3} + c_1x + c_2y + c_3$$

for $0 \le x \le l$, $-a \le y \le b$, where $l, a, b > 0$, $m \ge 0$, and c_i are arbitrary constants. If the vertical component of the displacement vector satisfies conditions (13) and the boundary conditions $\zeta(x,-a) = \zeta(x,b) = 0$, $\zeta(0,y) = \zeta(l,y)$, then the surface S will be rigid if and only if (14) holds.

We note that in our case conditions (13) coincide with the so-called "conjugacy conditions" ($\mathbf{U}^+ = \mathbf{U}^-$ for $y = 0$) in the theory of infinitely small bendings.

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