Reports of the Academy of Sciences of the USSR

1958

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1958. Vol. 119, No. 2

THEORY OF ELASTICITY

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ON SUBHARMONIC OSCILLATIONS OF RODS WITH NONLINEAR INERTIA

(Presented by Academician N. N. Bogolyubov, 25 IX 1957)

The note is devoted to subharmonic solutions of a nonlinear equation describing transverse oscillations of an elastic rod with allowance for axial inertial forces caused by the presence of a concentrated mass on a movable support. Free and forced oscillations with allowance for external action were considered in papers ([1–4]). In contrast to the equation of a linear oscillator, in the case considered, under certain conditions, ultraharmonic and subharmonic oscillations may exist in addition to the harmonic motion predicted by the linear interpretation of the problem. Experimental investigation shows that under some conditions, in addition to the well-known small oscillations with the period of the disturbing force, oscillations of considerably larger amplitude may become established, with a period that is an integral multiple of the period of the external force.

1. Let us consider transverse oscillations of a rod of constant cross-section with hinged ends. Let the movable support carry a concentrated mass M. Denote by m the mass per unit length, and by l the span. Under the action of a concentrated pulsating force $P\cos pt$, applied in the middle of the span, taking the displacement of an arbitrary point of the elastic line in the form $y = g(t)\sin \pi x/l$, we obtain the equation

$$q'' + 2\varepsilon q' + \varkappa q(q^2)'' + \omega^2 q = S\cos pt. \tag{1}$$

Here ε is the coefficient of viscous damping, $\varkappa$ is the coefficient of nonlinear inertia, $\varkappa = M\pi^4/4ml^3$, $S = 2P/ml$.

We seek a solution of (1) in the form

$$q = C + A_{1/n}\cos(pt/n + \varphi) + A_1\cos(pt + \psi). \tag{2}$$

To determine the constants we use B. G. Galerkin' s method. For this purpose we substitute (2) into (1), after which, expanding the result in a Fourier series and equating to zero the coefficients of the terms with trigonometric functions of the basic frequencies 0, p/n, and p, we find the necessary equations.

If the result of substituting (2) into (1) does not contain combination frequencies coinciding with the basic ones, then for a dissipative system ($\varepsilon \neq 0$) the relations $A_{1/n} = 0$, $C = 0$, $q = A_1 \cos(pt + \psi)$ will hold, where

$$p = \sqrt{\frac{\omega^2 - S/A_1}{1 + \varkappa A_1^2}}.$$

Subharmonic (demultiplicational) resonance occurs in the presence of combination tones coinciding with the basic ones, and since

the result of substituting (2) into (1) contains terms with fundamental frequencies 0, p/n, and p, and combination tones $2p$, $2p/n$, $3p$, $3p/n$, $|p/n \pm p|$, $|2p/n \pm p|$, $|2p \pm p/n|$; thus in our case of biharmonic oscillations this is possible for $n = 2$ and $n = 3$.

2. Subharmonic resonance of order $1/2$ ($n = 2$)

In this case we have the system of equations

$$C\left[\omega^2 - \varkappa\left(\frac{1}{4}A_{1/2}^2 p^2 + A_1^2 p^2\right)\right] - \frac{3}{8}\varkappa p^2 A_{1/2}^2 A_1 \cos(2\varphi - \psi) = 0,$$

$$A_{1/2}\left\{\omega^2 - \frac{1}{4}p^2 - \frac{1}{4}\varkappa p^2\left[2C^2 + A_{1/2}^2 + 5A_1^2 + 6CA_1\cos(2\varphi - \psi)\right]\right\} = 0,$$

$$A_{1/2}\left[-\varepsilon p - CA_1\frac{3}{2}\varkappa p^2 \sin(2\varphi - \psi)\right] = 0, \tag{3}$$

$$A_1\left[\omega^2 - p^2 - \varkappa p^2\left(2C^2 + \frac{5}{4}A_{1/2}^2 + A_1^2\right)\right] - \frac{3}{4}\varkappa C p^2 A_{1/2}^2 \cos(2\varphi - \psi) = S\cos\psi,$$

$$2\varepsilon p A_1 - \frac{3}{4}\varkappa C p^2 A_{1/2}^2 \sin(2\varphi - \psi) = -S\sin\psi.$$

In what follows we shall consider the system without friction. Here two cases may occur: either $C = 0$, or $2\varphi - \psi$ is equal to 0 or π.

1) $C = 0$. We shall assume that $\psi = 0$, while the amplitudes of the harmonic component A_1 and of the subharmonic $A_{1/2}$ may change sign. Then from (3) we obtain:

$$p = 2\sqrt{\frac{4\omega^2 + S/A_1}{1 + 21\varkappa A_1^2}}, \tag{4}$$

$$A_{1/2} = \sqrt{\frac{4\omega^2 - p^2}{\varkappa p^2} - 5A_1^2}. \tag{5}$$

In this form the system admits the following course of solution: assign values of A_1; from (4) determine p, and then $A_{1/2}$ from (5). Formula (5) shows that if $p > 2\omega$, then demultiplication resonance is impossible. For $A_1 = -S/4\omega^2$, $p_{\min} = 0$ and $A_{1/2} \to \infty$. From (5) we find that the maximum value of the frequency at which subharmonic oscillations are still possible is

$$p_{\max} = 2\omega/\sqrt{1 + 5\varkappa A_1^2}. \tag{6}$$

The solution (4), (5) has a common point with the harmonic one—the bifurcation point. For $A_{1/2} = 0$ we write

$$\omega^2 - \frac{1}{4}p^2 - \frac{5}{4}\varkappa p^2 A_1^2 = 0, \qquad A_1(\omega^2 - p^2 - \varkappa p^2 A_1^2) = S. \tag{7}$$

As is known (2), the second equation (7) is the equation of the curve $A_1(p)$ in the case of the harmonic solution. These two equations determine one branching point in the p–A_1 plane.

Formula (5) shows that the oscillations under consideration are sufficiently close to free oscillations with a frequency equal to one half of the frequency of the external force, for sufficiently small S ($A_1 \ll A_{1/2}$). Figure 1 shows the curves $A_1(p)$ and $A_{1/2}(p)$ for the parameter values $\omega^2 = 2$, $\varkappa = 0.02$, $S = 2$, and the curve for the cosine approximation ($A_{1/2} = 0$).

2) $2\varphi - \psi$ is equal to zero or π. System (3) becomes the following:

$$\begin{gathered}
C\left[\omega^2 - \varkappa\left(\frac{1}{4}A_{1/2}^2 p^2 + A_1^2 p^2\right)\right] \mp \frac{3}{8}\varkappa p^2 A_{1/2}^2 A_1 = 0, \\
\omega^2 - \frac{1}{4}p^2 - \frac{1}{4}\varkappa p^2(2C^2 + A_{1/2}^2 + 5A_1^2 \pm 6CA_1) = 0, \qquad (8) \\
A_1\left[\omega^2 - p^2 - \varkappa p^2\left(2C^2 + \frac{5}{4}A_{1/2}^2 + A_1^2\right)\right] \mp \frac{3}{4}\varkappa C p^2 A_{1/2}^2 = S.
\end{gathered}$$

System (8) was solved by selection; starting from the roots of the system for $C = 0$, the roots of equations (8) were found by subsequent refinement. In this process it turned out that the quantity C is in fact small and the resonance cur-

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

ones constructed for this system are, in character, close to the case $C = 0$. The solutions (8) and (7) have one common bifurcation point.

3. **Subharmonic resonance of order** $1/3$ $(n = 3)$**.** In this case there are no combination frequencies tending to zero. Taking $\varphi = \psi = 0$, and assuming that the quantities $A_{1/3}$ and A_1 may be either positive or negative, we find

$$\omega^2 - \frac{1}{9}p^2 - \frac{1}{9}\varkappa p^2\left(A_{1/3}^2 + 10A_1^2 + 3A_{1/3}A_1\right) = 0; \tag{9}$$

$$A_{1/3}^3\varkappa\omega^2\left(-1 - 9\beta + 3\beta^2 + \beta^3\right) - A_{1/3}^2\varkappa S\left(1 + 3\beta + 10\beta^2\right) - A_{1/3}8\omega^2\beta - S = 0, \tag{10}$$

where $\beta = A_1/A_{1/3}$.

Fig. 1

Fig. 2

In this form equations (9) and (10) admit the following procedure for solution: we prescribe values of β from (10), find $A_{1/3}$, and then p from (9). In Fig. 2 the resonance curves are plotted for the parameter values $\omega^2 = 2$, $\varkappa = 0.02$, $S = 2$, with the upper curve $A_{1/3}(p)$ corresponding to the lower curve $A_1(p)$. The curve for $A_{1/3} = 0$ represents the harmonic solution.

Analyzing the system (9)–(10), it is essential to note the following properties of the solution:

1) Subharmonic oscillations of order 1/3 are impossible if the frequency of the disturbing force is higher than three times the natural frequency of the linear oscillations,

$$p_{\max} = \frac{3\omega}{\sqrt{1 + {}^{31}/_{4}\,\varkappa A_1^2}}.$$

2) The solution has one common point with the harmonic one, which is a branching point and is determined by the equations

$$\omega^2 - \frac{1}{9}p^2 - \frac{10}{9}\varkappa p^2 A_1^2 = 0, \qquad A_1(\omega^2 - p^2) - \varkappa p^2 A_1^3 = S. \tag{11}$$

Fig. 3

Figure 3: Fig. 3

Fig. 4

Figure 4: Fig. 4

3) Oscillations for a sufficiently small value of A_1 are close to free oscillations with frequency $p/3$. It is characteristic that for those values of the frequency p for which $A_1 = 0$, the obtained solution $q = A_{1/3} \cos(pt/3)$ is exact.

4. We now proceed to the description of the experiment. The setup is shown schematically in Fig. 3. Here 1—beams, 2—lever, 3—load of mass M_0, 4—spring of stiffness c_0, 5—oscillation exciter, 6—wire gauges. Owing to the fact that it was possible to vary both the height of attachment of the load and its magnitude, it was possible to vary the values of the parameters

within the required limits. To create the exciting force acting only in the vertical direction, a centrifugal vibrator was used. The oscillations were recorded by means of a loop oscillograph after amplification onto photographic film. To count the number of revolutions, a contact breaker was used; in this case, on the film, one revolution of the shaft corresponded to a dash and a gap. To mark time, a loop connected through a rheostat to industrial current with a frequency of 50 Hz was used. The structural implementation of the scheme did not make it possible to eliminate elastic connections, although the predominant influence, for the selected parameter values, was exerted by the inertial forces, since the influence of the parameter $\varkappa$ is greater the larger p^2 is.

Fig. 3

Figure 4 shows an oscillogram recorded for subharmonic oscillations of order 1/2. As can be seen from Fig. 4, under subharmonic resonance the system makes one oscillation for two revolutions of the shaft, i.e., the resulting motion has a period equal to twice the period of the external force.

Fig. 4

The quantitative comparison was carried out with simultaneous allowance for the forces of elasticity and inertia. In this case, good agreement of the results was obtained.

In conclusion, we note that demultiplication resonance for a mechanical system described by equation (1) is apparently being observed for the first time.

Received
18 IX 1957

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Note: Figure translations are in progress. See original paper for figures.