ON THE QUESTION OF FORCED PSEUDOHARMONIC VIBRATIONS OF RODS WITH ELASTICALLY YIELDING SUPPORTS

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Abstract

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V. A. IVOVICH

ON THE QUESTION OF FORCED PSEUDO-HARMONIC VIBRATIONS OF RODS WITH ELASTICALLY YIELDING SUPPORTS

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In the present note we consider the equation [3]

$$\frac{d^2q}{dt^2} + \omega^2 q + \gamma q^3 + \varkappa q \frac{d^2}{dt^2}(q^2) = \frac{p_0}{m}\cos\Omega t, \tag{1}$$

which describes the vibratory motion of a hinged-supported rod disturbed by the force

$$p = p_0 \sin\frac{\pi x}{l}\cos\Omega t$$

under the assumption that one of the supports possesses elastic compliance. Here the amplitudes are assumed to be small, but finite.

The coefficients γ and $\varkappa$ in (1) characterize, respectively, the axial elastic and inertial forces. Although the solution of this equation in elementary functions is unknown, the literature gives methods that in principle make it possible to obtain the solution with arbitrary accuracy. Most methods require the constants γ and $\varkappa$ to be small; moreover, obtaining higher approximations becomes very complicated.

The method applied here consists in replacing the harmonic external force by a force whose law of variation in time is similar to the form of the natural vibrations; then equation (1) admits an exact solution. A special feature of the problem is the circumstance that, for small values of the nonlinear terms, the chosen approximating disturbing-force function is close in its character of variation to a trigonometric one. Therefore the approximate solution obtained in this way for the case of harmonic excitation proves sufficiently accurate for practical purposes both in the neighborhood of free vibrations for all values of the amplitudes and at small amplitudes.

We replace equation (1) by the equation

$$\frac{d^2q}{dt^2} + \omega^2 q + \gamma q^3 + \varkappa q \frac{d^2}{dt^2}(q^2) = \frac{p_0}{mA} q. \tag{2}$$

Introducing the notation

$$a = A^2, \qquad b = 0, \qquad c = -\frac{1}{2\varkappa}, \qquad d = -\frac{2(\omega^2 - p_0/mA) + \gamma A^2}{\gamma}, \tag{3}$$

and assuming that $a > q^2 > b > c > d$, we integrate equation (2). The period of vibrations T is written in the form

$$T = 4\sqrt{\frac{\varkappa}{\gamma}} \int_0^a \frac{\sqrt{\tau - c}\, d\tau}{\sqrt{(a-\tau)(\tau-b)(\tau-d)}},$$

where $\tau = q^2$. Replacing the variable τ by φ from the relation

$$\sin^2\varphi = \frac{(b-d)(a-\tau)}{(a-b)(\tau-d)}$$

gives

$$T = 8\sqrt{\frac{\chi}{\gamma(a-c)(b-d)}} \left[(d-c)K(k) + (a-d)\Pi(\alpha^2, k)\right]. \tag{4}$$

Here $K(k)$ and $\Pi(\alpha^2, k)$ are complete elliptic integrals of the first and third kinds with modulus k and parameter α^2,

$$k = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}, \qquad \alpha^2 = \frac{b-a}{b-d} < 0.$$

Let us express $\Pi(\alpha^2, k)$ in terms of $\Lambda_0(\psi, k)$—Heuman' s function ([1]), using the well-known formula from the theory of elliptic integrals ([2])

$$\Pi(\alpha^2, k) = \frac{k^2 K(k)}{k^2 - \alpha^2} - \frac{\pi \alpha^2 \Lambda_0(\psi, k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2 - k^2)}}, \tag{5}$$

where

$$\psi = \arcsin\sqrt{\frac{\alpha^2}{\alpha^2 - k^2}}; \tag{6}$$

$$\Lambda_0(\psi,k) = -\frac{2\sin\psi\sqrt{1+k^2\tan^2\psi}}{\pi}\int_0^{K(k)} \frac{\operatorname{dn}^2 u\,du}{1+k^2\tan^2\psi\,\operatorname{sn}^2 u}; \tag{7}$$

$$u = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}; \tag{8}$$

$\operatorname{dn} u$, $\operatorname{sn} u$ are Jacobi elliptic functions.

Substitution of (5) into (4), after some transformations, gives for the circular frequency Ω the dependence

$$\Omega = -\frac{1}{2\sqrt{\chi/\gamma}\,\Lambda_0(\psi,k)},$$

$$\psi = \arcsin\sqrt{\frac{\gamma(1+2\chi A^2)}{4\chi(\omega^2 - p_0/mA + \gamma A^2)}}, \tag{9}$$

$$k = \sqrt{\frac{A^2\{2\chi[2(\omega^2 - p_0/mA) + \gamma A^2] - \gamma\}}{(1+2\chi A^2)[2(\omega^2 - p_0/mA) + \gamma A^2]}}.$$

This solution has a real value of the modulus k when

$$\gamma < 2\chi\left[2\left(\omega^2 - \frac{p_0}{mA}\right) + \gamma A^2\right].$$

Let us now suppose that

$$\gamma > 2\chi\left[2\left(\omega^2 - \frac{p_0}{mA}\right) + \gamma A^2\right].$$

Carrying out the integration (2), in this case we find for the frequency Ω the dependence

$$\Omega = \frac{\pi}{2\left\{\frac{\gamma - 2\chi[2(\omega^2 - p_0/mA) + \gamma A^2]}{\gamma\sqrt{\omega^2 + \gamma A^2 - p_0/mA}}\right\} K(k) - 2\pi\sqrt{\frac{\chi}{\gamma}}\,\Lambda_0(\psi,k)}, \tag{10}$$

where

$$k = \sqrt{\frac{A^2\{\gamma - 2\chi[2(\omega^2 - p_0/mA) + \gamma A^2]\}}{2(\omega^2 + \gamma A^2 - p_0/mA)}},$$

Figure 1

Figure 1: Figure 1

Figure 2

Figure 2: Figure 2

$$\psi = \arcsin \sqrt{\frac{4\chi(\omega^2 + \gamma A^2 - p_0/mA)}{\gamma(1 + 2\chi A^2)}}.$$

Figures 1 and 2 show the resonance curves for the cases of the predominant influence of nonlinear inertia and elasticity. As can be seen, these

curves differ not only from the resonance curve of the linear problem, but also from the resonance curves in the case where only one nonlinear factor is present.

Fig. 1. Resonance curve in the case of the predominant influence of nonlinear inertance

Fig. 2. Resonance curve in the case of the predominant influence of nonlinear elasticity

Let us briefly consider two particular assumptions.

1. $\gamma = 0,\ \varkappa \neq 0$. From (9) we find

$$\left(\frac{2E(k)\,\Omega}{\pi}\right)^2 = \frac{\omega^2 - p_0/mA}{1 + 2\varkappa A^2}, \tag{11}$$

where

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\varphi}\, d\varphi, \qquad k^2 = \frac{2\varkappa A^2}{1 + 2\varkappa A^2}.$$

The time function $q = A\cos\varphi$, while the angle φ is determined from the relation

$$\frac{2E(k)}{\pi}\,\Omega t = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2\varphi}\, d\varphi \qquad (0 < k < 1). \tag{12}$$

In problems of a practical character, the magnitude of the modulus $k \ll 1$; moreover, the character of the variation of $\cos\varphi$ is very close to the character of the variation of $\cos\Omega t$, and for $k = 0$, $\cos\varphi = \cos\Omega t$.

II. $x = 0,\ \gamma \neq 0$. The time function q in this case is expressed through the elliptic cosine in the form $q = A\ \mathrm{cn}\left(\frac{2K(k)}{\pi}\Omega t, k\right)$.

Fig. 3

Figure 3: Fig. 3

Fig. 4

Figure 4: Fig. 4

Expression (10) is transformed into the form:

$$\Omega = \frac{\pi}{2K(k)}\sqrt{\omega^2 + \gamma A^2 - \frac{p_0}{mA}}, \tag{13}$$

where

$$k^2 = \frac{\gamma A^2}{2\left(\omega^2 + \gamma A^2 - p_0/mA\right)}.$$

Fig. 3

Fig. 4

For comparison with the results of the approximate theory, let us write, with accuracy up to terms k^2,

$$E(k) = \frac{\pi}{2}\left(1 - \frac{k^2}{4}\right); \tag{14}$$

$$\left(\frac{2K(k)}{\pi}\right)^2 = 1 + \frac{k^2}{2}. \tag{15}$$

Using expressions (11) and (14), (13) and (15), we obtain formulas known in the literature [(3)]:

$$\Omega = \sqrt{\frac{\omega^2 - p_0/mA}{1 + \chi A^2}}, \tag{16}$$

$$\Omega = \sqrt{\omega^2 + \frac{3}{4}\gamma A^2 - \frac{p_0}{mA}}. \tag{17}$$

The proposed method does not require the construction of higher approximations, and since, for small values of the modulus k, the general character of the variations of the time function $q(t)$ is close to a harmonic law, for small values of the amplitudes a sufficiently accurate solution is obtained.

If the amplitude value of the disturbing force is small, then this method gives a good approximation in the neighborhood of free oscillations for all values of the oscillation amplitudes.

In Figs. 3 and 4, graphs are given of the variation of $\Lambda_0(\psi, k)$ as a function of the argument ψ and the modulus k.

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Note: Figure translations are in progress. See original paper for figures.