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Abstract

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MATHEMATICS

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RIEMANNIAN SPACES OF CURVATURE BOUNDED BELOW BY A POSITIVE NUM- BER

(Presented by Academician S. L. Sobolev on 11 XII 1957)

In our previous note ⁽¹⁾, for m -dimensional twice continuously differentiable spaces R_k^m with a complete metric, whose curvature is bounded below by the number k , the following theorem was formulated:

Theorem 1. In R_k^m , the angles of an arbitrary triangle ABC composed of shortest arcs are not smaller than the corresponding angles of the triangle $(ABC)_k$ in the plane R_k of constant curvature k with the same side lengths.

For the case $k = 0$, the proof of Theorem 1 is given in ⁽¹⁾. For the case when $k < 0$ or $k > 0$, but $AB + AC + BC < 2\pi/\sqrt{k}$, the proof of Theorem 1 is obtained by a simple modification of the proof for $k = 0$.

In the present article the case $k > 0$ is considered. In this case, for the triangle ABC , two possibilities remain to be considered:

I. $AB + AC + BC = 2\pi/\sqrt{k}$.

II. $AB + AC + BC > 2\pi/\sqrt{k}$.

It is proved that in R_k^m ($k > 0$) case II is impossible; for case I, Theorem 1 is proved, which thereby is established in full. At the same time, a number of other geometric results are obtained.

Since Theorem 1 is valid for the case $AB + AC + BC < 2\pi/\sqrt{k}$, by passing to the limit we obtain:

Theorem 2. If in the space R_k^m there exists a triangle the sum of whose side lengths is equal to $2\pi/\sqrt{k}$, and none of the sides is equal to the sum of the other two, then all angles of this triangle are equal to π , so that the triangle is a closed shortest curve.

Theorem 3. In the space R_k^m , the sum of the side lengths of a triangle does not exceed $2\pi/\sqrt{k}$.

Proof. Suppose that the theorem is false. Then there exists a triangle ABC for which $AB + AC + BC > 2\pi/\sqrt{k}$. Let $X(t)$ and $Y(t)$ be variable points of AB and AC ($0 \leq t \leq 1$); there exists a set M of values of t for which

$$AX(t) + AY(t) + X(t)Y(t) = 2\pi/\sqrt{k}.$$

Two cases are possible:

I. There is at least one $t = t_0 \in M$ for which $X(t_0)Y(t_0) < \pi/\sqrt{k}$.

II. For every $t \in M$, $X(t)Y(t) = \pi/\sqrt{k}$.

The first case is impossible by virtue of Theorem 2, the non-branching condition for shortest arcs, and our assumptions.

In the second case M contains only one value $t = t_1$. Hence

for $t > t_1$,

$$AX(t) + AY(t) + X(t)Y(t) > 2\pi/\sqrt{k}.$$

Using Theorem 1 for the case where the sum of the side lengths of the triangles is strictly less than $2\pi/\sqrt{k}$, and the nonbranching condition for shortest arcs, by passage to the limit one can prove that, for $t > t_1$, the lines $AX(t)Y(t)A$ are closed geodesics. Since the closed geodesics $AX(t)Y(t)A$ for $t > t_1$ have the common part $X(t_1)AY(t_1)$, they all coincide. In particular, $AX(1)Y(1)A$ coincides with $AX(t)Y(t)A$ for any $t > t_1$. But for values $t > t_1$ sufficiently close to t_1 , the length of $AX(t)Y(t)A$ differs arbitrarily little from $2\pi/\sqrt{k}$. Hence the length of the line $ABCA$ is equal to $2\pi/\sqrt{k}$. The contradiction obtained proves the theorem.

From Theorem 3 it follows easily:

Theorem 4. In the space R_k^m there exists no shortest arc of length greater than π/\sqrt{k} .

Theorem 4, unlike the other theorems of this note, can also be proved by classical methods (with the aid of the Jacobi equations).

If in the space R_k^m ($k > 0$) a triangle ABC is given for which

$$AB + AC + BC = 2\pi/\sqrt{k}$$

and one of the sides is equal to π/\sqrt{k} , then the corresponding triangle $(ABC)'_k$ of the plane of constant curvature k is not uniquely determined. In this case we agree to understand by the triangle $(ABC)'_k$ the degenerate triangle, i.e., the triangle in which two angles are equal to zero. With this convention in mind, one can easily derive Theorem 1 in full from Theorem 2.

Theorem 5. *If in the space R_k^m there exists a shortest arc of length π/\sqrt{k} , then R_k^m is the m -dimensional sphere of radius $1/\sqrt{k}$.*

Proof. Let the length of the shortest arc AB be π/\sqrt{k} . From Theorems 3 and 2 it follows easily that any geodesic p passing through the point A contains B and is a closed geodesic of length $2\pi/\sqrt{k}$.

Let $X(t)$ be a variable point of p , where t is the arc length, and let

$$X(0) = A;$$

then

$$X(\pi/\sqrt{k}) = B$$

and

$$X(t) = X(t + 2\pi/\sqrt{k}) \quad (0 \leq t < \infty).$$

Using Sturm's comparison theorem, one can prove that the curvature of the space R_k^m at any point of the geodesic p and in any two-dimensional direction tangent to p is equal to k , and that the nearest point on p conjugate to $X(t)$ is $X(t + \pi/\sqrt{k})$.

Moreover, we shall now prove that the arc

$$l = X(t)X(t + \pi/\sqrt{k})$$

of the geodesic p is a shortest arc for every t . Indeed, suppose that l is not a shortest arc. Then there is a greatest value $t = t_0$, $t < t_0 < t + \pi/\sqrt{k}$, such that the arc

$$l' = X(t)X(t_0)$$

of the geodesic p is a shortest arc. Surround the arc

$$X(t - \varepsilon)X(t + \pi/\sqrt{k} - \varepsilon)$$

by the central field of extremals \mathfrak{A} with center at the point $X(t - \varepsilon)$, where ε is a sufficiently small number. Then the shortest arcs

$$X(t)X(t + s),$$

where $s = t_0 - t + \sigma$ and σ is sufficiently small, do not belong to the field of extremals \mathfrak{A} . Therefore the limiting shortest arc

$$\overline{X(t)X(t_0)}$$

of the shortest arcs

$$X(t)X(t + s)$$

also does not belong to the field \mathfrak{A} , and consequently does not coincide with the arc l' .

On the other hand, applying Theorem 2 to the triangle

$$\overline{X(t)X(t_0)B}$$

(or to the triangle $X(t)X(t_0)A$), we are convinced that the angle between $\overline{X(t)X(t_0)}$ and $X(t)B$ is equal to π ; therefore, by the nonbranching condition for shortest arcs, $\overline{X(t)X(t_0)}$ coincides with l' . Thus, l is a shortest arc. Applying the preceding considerations to the shortest arc l (replacing the points A and B , respectively, by the points $X(t)$ and $X(t + \pi/\sqrt{k})$), we prove that the curvature of the space R_k^m at any point and in any two-dimensional direction is equal to k . Theorem 5 is proved, since it is easy to see that R_k^m is homeomorphic to a sphere.

Theorem 5 admits the following generalization.

Theorem 6. *If in the space R_k^m there exists a triangle ABC the sum of whose side lengths is equal to $2\pi/\sqrt{k}$, then R_k^m is an m -dimensional sphere of radius $1/\sqrt{k}$.*

Proof. By Theorem 5, it is enough to prove that in R_k^m there exists a shortest curve of length π/\sqrt{k} . Therefore we may assume that none of the sides of the triangle ABC is equal to the sum of the other two. Take on the shortest curve BC a point D such that $ABD = ACD = \pi/\sqrt{k}$. Join the point A to the point D by a shortest curve AD . We shall prove that $AD = \pi/\sqrt{k}$. Indeed, if this is not so, then the sums of the side lengths of the triangles ABD and ADC are less than $2\pi/\sqrt{k}$. Hence it follows that

$$\text{angle } A'B'D' < \pi; \quad \text{angle } A'C'D' < \pi, \quad (1)$$

where $A'B'D'$ is the angle of the triangle $(ABD)'_k$, and the angle $A'C'D'$ is the angle of the triangle $(ACD)'_k$.

Construct the quadrilateral $A'B'C'D'$, fitting together along equal sides the triangles $(ABD)'_k$ and $(ACD)'_k$. In the quadrilateral so obtained, the angle D' , by Theorem 1, is not greater than π ; on the other hand, D' cannot be equal to π , since then, by Theorem 2, all the remaining angles of the quadrilateral would also have to be equal to π , contrary to (1). Thus the quadrilateral $A'B'C'D'$ has three angles B' , C' , and D' less than π ; but from simple geometric considerations it follows easily that in the plane R_k there do not exist convex quadrilaterals the sum of whose side lengths is equal to $2\pi/\sqrt{k}$ and in which three angles are less than π . The contradiction obtained proves the theorem.

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CITED LITERATURE

1. V. A. **Toponogov**, DAN, **115**, 672 (1957).

Note: Figure translations are in progress. See original paper for figures.

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