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# MATHEMATICS

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Abstract

Full Text

## MATHEMATICS

Yu. Ya. Pogodin, V. A. Suchkov, and N. N. Yanenko

### ON TRAVELING WAVES OF THE EQUATIONS OF GAS DYNAMICS

(Presented by Academician A. D. Sakharov on 21 IX 1957)

In the work <sup>(1)</sup>, simple traveling waves of a system of quasilinear equations of the form

$$a_{ijk}(u_1, \dots, u_m) \frac{\partial u_i}{\partial x_k} = 0, \quad i, j, k = 1, \dots, m. \quad (1'')$$

were considered.

We shall call a solution  $u_i = u_i(x_1, \dots, x_m)$  of the system (1), satisfying  $m - r$  functional relations

$$\varphi_\alpha(u_1, \dots, u_m) = 0, \quad \alpha = 1, \dots, m - r,$$

a **traveling wave** of rank  $r$ . In this definition, a traveling wave of rank 1 coincides with the simple traveling wave of note <sup>(1)</sup>.

In the present work, by the example of the equations of gas dynamics of a polytropic gas, traveling waves of rank  $m - 1$  are considered.

The equations of motion of a polytropic gas

$$\rho \left( \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial p}{\partial x_i} = 0; \quad \frac{\partial \rho}{\partial t} + u_k \frac{\partial \rho}{\partial x_k} + \rho \frac{\partial u_k}{\partial x_k} = 0; \quad p = \frac{a^2 \rho^\gamma}{\gamma}, \quad i, k = 1, \dots, m-1 \quad (1')$$

in the adiabatic case, in the variables  $u_i$ ,  $\theta = \frac{a^2 \rho^{\gamma-1}}{\gamma - 1}$ , take the form

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial \theta}{\partial x_i} = 0; \quad \frac{\partial \theta}{\partial t} + u_k \frac{\partial \theta}{\partial x_k} + (\gamma - 1) \theta \frac{\partial u_k}{\partial x_k} = 0. \quad (2)$$

In the isothermal case  $a^2 = RT = \text{const}$ ,  $\gamma = 1$ ,  $\theta = \ln \rho$ , and instead of (2)

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + a^2 \frac{\partial \theta}{\partial x_i} = 0; \quad \frac{\partial \theta}{\partial t} + u_k \frac{\partial \theta}{\partial x_k} + \frac{\partial u_k}{\partial x_k} = 0. \quad (2')$$

For simplicity we restrict ourselves to the case  $m = 3$ , although the method of consideration is general and applicable not only to the case  $m = 3, 4$ , but also to systems of type (1). We seek the functional dependence in the form  $\theta = \varphi(u_1, u_2)$ .

Let the differential equation of the common level lines of the functions  $u_1, u_2, \theta$  be  $dx_i/dt = \Delta_i$ . Then for any function  $f(u_1, u_2, \theta)$  it is true that

$$\frac{\partial f}{\partial t} + \Delta_k \frac{\partial f}{\partial x_k} = 0.$$

Using this, we write equations (2) in the form

$$\begin{aligned} (u_1 - \Delta_1 + \varphi_1) \frac{\partial u_1}{\partial x_1} + (u_2 - \Delta_2) \frac{\partial u_1}{\partial x_2} + \varphi_2 \frac{\partial u_2}{\partial x_1} + 0 \cdot \frac{\partial u_2}{\partial x_2} &= 0; \\ 0 \cdot \frac{\partial u_1}{\partial x_1} + \varphi_1 \frac{\partial u_1}{\partial x_2} + (u_1 - \Delta_1) \frac{\partial u_2}{\partial x_1} + (u_2 - \Delta_2 + \varphi_2) \frac{\partial u_2}{\partial x_2} &= 0; \end{aligned} \quad (3)$$

$$\begin{aligned} [(\gamma - 1)\varphi + \varphi_1(u_1 - \Delta_1)] \frac{\partial u_1}{\partial x_1} + \varphi_1(u_2 - \Delta_2) \frac{\partial u_1}{\partial x_1} + \varphi_2(u_1 - \Delta_1) \frac{\partial u_2}{\partial x_1} + \\ + [(\gamma - 1)\varphi + \varphi_2(u_2 - \Delta_2)] \frac{\partial u_2}{\partial x_2} = 0, \end{aligned}$$

where

$$\varphi_i = \partial \varphi / \partial u_i, \quad i = 1, 2.$$

Let us require that, for a fixed function  $\varphi(u_1, u_2)$ , the traveling wave have the arbitrariness of two functions of one argument. For this it is necessary that the rank of system (3) be equal to 2.

Hence we obtain two cases:

- a)  $\varphi_i = \Delta_i - u_i = -a_i, \quad i = 1, 2;$
- b)  $a_1 \varphi_1 + a_2 \varphi_2 = 0, \quad (a_1^2 + a_2^2)[\varphi_1^2 + \varphi_2^2 - (\gamma - 1)\varphi] = (\gamma - 1)\varphi(\varphi_1^2 + \varphi_2^2).$

We shall restrict ourselves to the first case, which is the principal one. From a) it follows that the level lines are straight and  $\Delta_i = \partial \Delta / \partial u_i, \Delta = \varphi + \frac{1}{2}(u_1^2 + u_2^2)$ . Equations (3) take the form:

$$L^s = a_{ij}^s \frac{\partial u_i}{\partial x_j} = 0, \quad i, j, s = 1, 2; \quad (4)$$

$$a_{11}^1 = a_{22}^1 = 0, \quad a_{12}^1 = -a_{21}^1 = 1, \quad a_{ij}^2 = (\gamma - 1)\varphi\delta_{ij} - \varphi_i\varphi_j, \quad (5)$$

$\delta_{ij}$  is the Kronecker symbol. From (4), (5) it follows that the motion is potential. The relations  $\partial L^s / \partial t = 0$ , together with (5), give new conditions:

$$a_{ij}^s \Delta_{kl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} = 0, \quad \Delta_{kl} = \frac{\partial^2 \Delta}{\partial u_k \partial u_l}. \quad (6)$$

In order that (6) follow from (4), it is necessary and sufficient that

$$\begin{aligned} L(\varphi) = [(\gamma - 1)\varphi - \varphi_2^2](\varphi_{11} + 1) + 2\varphi_1\varphi_2\varphi_{12} + \\ + [(\gamma - 1)\varphi - \varphi_1^2](\varphi_{22} + 1) = 0. \end{aligned} \quad (7)$$

The conditions  $\partial^k L^s / \partial t^k = 0$ ,  $k > 1$ , give nothing new. Hence it follows:

**Theorem 1.** *If the function  $\varphi(u_1, u_2)$  satisfies equation (7), then the traveling wave has the arbitrariness of two functions of one argument.*

By means of the hodograph transformation, system (4) can be reduced to the form

$$[(\gamma - 1)\varphi - \varphi_1^2] \frac{\partial^2 X}{\partial u_2^2} + 2\varphi_1\varphi_2 \frac{\partial^2 X}{\partial u_1 \partial u_2} + [(\gamma - 1)\varphi - \varphi_2^2] \frac{\partial^2 X}{\partial u_1^2} = 0, \quad (8)$$

$$dX = x_1 du_1 + x_2 du_2.$$

For isothermal motion with  $a^2 = 1$ ,  $\varphi, X$  satisfy the equations

$$(1 - \varphi_1^2)(\varphi_{22} + 1) + 2\varphi_1\varphi_2\varphi_{12} + (1 - \varphi_2^2)(\varphi_{11} + 1) = 0; \quad (9)$$

$$(1 - \varphi_1^2) \frac{\partial^2 X}{\partial u_2^2} + 2\varphi_1\varphi_2 \frac{\partial^2 X}{\partial u_1 \partial u_2} + (1 - \varphi_2^2) \frac{\partial^2 X}{\partial u_1^2} = 0. \quad (10)$$

**Theorem 2.** *In the case when  $\varphi = C_0 + C_1 u_1 + C_2 u_2 - \frac{u_1^2 + u_2^2}{2}$ ,  $C_i$  are constants, all flows are conical, i.e., the straight level lines meet at one point (a centered traveling wave of rank 2). For the remaining solutions  $\varphi$  of equation (7) the motions are, generally speaking, not conical, but for any solution  $\varphi$  there is a solution  $X$  of equation (8) determining a conical flow.*

Fig. 1

Figure 1: Fig. 1

We shall now solve the problem of two plane pistons, restricting ourselves, for simplicity, to the isothermal case, in which the solution is obtained in explicit form.

Let the planes  $x_1 = 0$ ,  $x_2 = 0$  bound a mass of polytropic gas contained in the interior angle. Starting from  $t = 0$ , the planes move according to the law  $x_i = f_i(t)$ . At each instant in the  $x_1, x_2$ -plane we shall have the following regions of motion (see Fig. 1): in region *I*, gas at rest; in regions *IIa*, *IIb*, simple traveling waves. In region *III* we shall seek the solution in the form of a traveling wave of rank 2. From the condition of continuous matching of wave *III* to waves *IIa*, *IIb* it follows that the solution of equation (9) satisfies the boundary conditions

$$\varphi(u_1, 0) = u_1 + \theta_0, \quad \varphi(0, u_2) = u_2 + \theta_0. \quad (11)$$

It is not difficult to see that equation (9) and conditions (11) are satisfied by the function  $\theta = \varphi(u_1, u_2) = u_1 + u_2 + \theta_0$ . From equation (10) it then follows that  $u_i = g_i(x_i, t)$ . It is not difficult to see that the fronts of weak discontinuities will be straight lines parallel to the axes. The argument remains valid also in the case when one or both fronts are shock waves moving with constant velocity. Thus, the solution has the form:

in region *I*,  $u_1 = u_2 = 0$  (region of rest);

in region *IIa*,  $u_1 = g_1(x_1, t)$ ,  $u_2 = 0$  (simple traveling wave);

in region *IIb*,  $u_1 = 0$ ,  $u_2 = g_2(x_2, t)$  (simple traveling wave);

in region *III*,  $u_1 = g_1(x_1, t)$ ,  $u_2 = g_2(x_2, t)$  (traveling wave of rank 2).

**Fig. 1**

In all regions  $\theta = \varphi(u_1, u_2) = u_1 + u_2 + \theta_0$ , if there are no strong discontinuities. Otherwise an additive constant depending on the region enters into the expression for  $\theta$ .

The functions  $g_i(x_i, t)$  are solutions of the Riemann equations

$$u_i = F_i[x_i - (u_i + 1)t].$$

It is clear in this case that

$$f'_i(t) = F_i[f_i(t) - (f'_i + 1)t].$$

The indicated representation is valid in the following cases:

1. The lines  $x_i = C_i t$  are fronts of weak discontinuity. Then  $C_1 = C_2 = 1$ , and the functions  $f_i(t)$  may be arbitrary.
2. The line  $x_1 = C_1 t$  is a front of weak discontinuity, and the line  $x_2 = C_2 t$  is a shock-wave front. Then  $C_2 > 1$ ,  $f_2'(t) = u_{10} > 0$ ,  $C_1 = 1$ ,  $f_1(t)$  is arbitrary. In *IIa*, *III* there are simple traveling waves; in *I*, *I Ib*, constant motions.
3. Both lines  $x_1 = C_1 t$ ,  $x_2 = C_2 t$  are fronts of shock waves. Then  $C_1 > 1$ ,  $C_2 > 1$ ,  $f_1' = u_{10} > 0$ ,  $f_2' = u_{20} > 0$ . In all regions there are constant motions.

In the adiabatic case the function  $\varphi$  is determined in the same way as in the isothermal case, as a solution of equation (7) with the corresponding boundary conditions; for example, in case 1

$$\varphi(u_1, 0) = \frac{1}{\gamma - 1} \left[ \frac{\gamma - 1}{2} u_1 + C_0 \right]^2; \quad \varphi(0, u_2) = \frac{1}{\gamma - 1} \left[ \frac{\gamma - 1}{2} u_2 + C_0 \right]^2.$$

However, in this case the fronts become curved, and the boundary-value problem for the function  $X$  becomes more complicated.

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## CITED LITERATURE

1. N. N. Yanenko, DAN, **109**, No. 1, 44 (1956).

*Note: Figure translations are in progress. See original paper for figures.*

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