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Abstract

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ON THE FORMS OF EXTREMAL MULTIPLY MONOTONE POLYNOMIALS

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In the present note (which belongs to the same circle of ideas as ([1])) we consider polynomials

$$y_n(x) = \sum_{k=0}^{n} p_k x^k \tag{1}$$

of degree $\leq n$ with real coefficients, belonging to $T_n^{(h)}$ or $B_n^{(h)}$, as well as functions of the class G_{2m} ($T_n^{(h)}$ is the class of polynomials (1), multiply monotone of order $h+1$ on the interval $[-1,+1]$; $B_n^{(h)}$ is the class of polynomials (1) whose derivatives of all odd orders up to and including the $(2h+1)$-st are nonnegative on the whole real axis, and whose derivatives of even orders up to and including the $2h$-th are nonnegative at the point $x=-1$; G_{2m} is the class of functions

$$f_{2m}(x) = \int_{-\infty}^{x} e^{-z^2} y_{2m}(z)\, dz,$$

where $y_{2m}(x)$ is a polynomial (1), nonnegative on the whole real axis).

Let the coefficients of the polynomials (1) be subject to several admissible (i.e., compatible and not contradicting the monotonicity under consideration) constraints of the form

$$\omega_j(y_n) = \sum_{k=0}^{n} \alpha_{kj} p_k = A_j, \tag{2}$$

where α_{kj} and A_j are given real numbers and at least one $A_j \neq 0$ ($j = 1, \ldots, s;\ s \leq n$).

A polynomial $y_n^*(x)$ of the form (1), whose coefficients are subject to the admissible constraints (2), will be called an **extremal polynomial on a certain interval** if on this interval it has the least possible oscillation L_n (as is known, the **oscillation** of a monotone polynomial on a certain interval is the difference between its values at the end and at the beginning of this interval; an extremal function $f_{2m}^*(x) \in G_{2m}$ is defined analogously). Then the following theorems hold.

Theorem 1. *If the coefficients of a polynomial $y_n(x) \in T_n^{(h)}$ are subject to one constraint (2) (where we assume that $y_n(-1) = 0$ if $\alpha_{01} \neq 0$), then among these polynomials there exists an extremal polynomial on $[-1, +1]$ of the form*

$$y_n^*(x) = \int_{-1}^{x} (x - z)^h (1 - z)^\alpha (1 + z)^\beta U_m^2(z)\, dz, \tag{3}$$

where $U_m(x)$ is some polynomial of degree $\leq m = \frac{1}{2}(n - 1 - h - \alpha - \beta)$, all

whose roots lie in the interval $[-1, +1]$; α, β are numbers equal to 0 or 1 and determined by the nature of the constraint.

Theorem 2. Let the polynomials $y_n(x) \in B_n^{(h)}$ and their coefficients be subject to two conditions (2) (we additionally assume that $y_n(-1) = 0$, if

$$\begin{vmatrix} \alpha_{01} & A_1 \\ \alpha_{02} & A_2 \end{vmatrix} = 0,$$

when at least one number $\alpha_{0j} \neq 0$). Then among these polynomials there is a polynomial extremal on $[-1, +1]$ of the form

$$y_n^*(x) = \int_{-1}^{x} (x - z)^{2h} U_m^2(z)\, dz, \tag{4}$$

where $U_m(x)$ is some polynomial of degree $\leq m = \frac{1}{2}(n - 1 - 2h)$ with real coefficients.

Theorem 3. If the function $f_{2m}(x) \in G_{2m}$, and the coefficients of the polynomial $y_{2m}(x)$ are subject to two constraints (2), then among these functions there exists a function extremal on the entire real axis of the form

$$f_{2m}^*(x) = \int_{-\infty}^{x} e^{-z^2} U_m^2(z)\, dz, \tag{5}$$

where $U_m(x)$ is some polynomial of degree $\leq m$ with real coefficients.

The question of uniqueness of the extremal polynomials (3), (4) and functions (5) is not considered in the present note (as also in (1)).

The theorems indicated above make it possible to solve a number of extremal problems; we indicate some of them.

Problem 1. Among the polynomials

$$y_n(x) = x^n + \sum_{k=1}^{n-1} p_{kx}^k \in B_n^{(h)}$$

find the one extremal on the interval $[-1, +1]$, if $y_n^{(2h+1)}(\xi) = A^2$ (ξ and $A^2 > 0$ are given real numbers). Also find the magnitude of the least oscillation L_n.

Problem 2. In the class G_{2m}, find the extremal function $f_{2m}^*(x)$, if it is known that $f'_{2m}(0) = A^2$, $f''_{2m}(0) = B$ ($A^2 > 0$ and B are given real numbers). Also find the magnitude of the least oscillation L_{2m}.

To solve these problems, using Theorems 2 and 3 and carrying out arguments analogous to those used in (1) (in Problem 1 we expand the polynomial in the normalized Jacobi polynomials $\hat{J}_k(x)$ with "weight" $(1-x)^{2h}$, in Problem 2 in the normalized Hermite polynomials $\hat{H}_k(x)$), we obtain the following results.

1. In Problem 1

$$y_n^*(x) = \int_{-1}^{x} (x-z)^{2h} \left[a_m \hat{J}_m(z) + b \, \frac{\sum_{k=0}^{m-1} \hat{J}_k(\xi) \hat{J}_k(z)}{\sum_{k=0}^{m-1} \hat{J}_k^2(\xi)} \right]^2 dz;$$

$$L_n = a_m^2 + \frac{(m+h)\sqrt{(2m+2h)^2 - 1}\, b^2}{m(m+2h) \left[\hat{J}'_m(\xi) \hat{J}_{m-1}(\xi) - \hat{J}'_{m-1}(\xi) \hat{J}_m(\xi) \right]}, \tag{6}$$

where

$$m = \frac{1}{2}(n - 1 - 2h); \qquad a_m^2 = 2^{2m+2h+1} \frac{m!^2 (m+2h)!^2}{(2h)!(2m)!(2m+2h)!};$$

$$b = \frac{A}{\sqrt{(2h)!}} - a_m \hat{J}_m(\xi) \quad \left(\text{the signs of } A \text{ and } a_m \text{ are chosen so that sign} \, \frac{A}{\sqrt{(2h)!}} = \operatorname{sign} a_m \hat{J}_m(\xi) \right);$$

obviously, $\mathcal{L} = \inf L_n$, which coincides with the results of paper [(2)] (Problem 1).

2. In Problem 2

$$f_{2m}^*(x) = \int_{-\infty}^{x} e^{-z^2} \left(\sum_{k=0}^{m} a_k H_k(z) \right)^2 dz,$$

where

$$a_{2l} = \pm \frac{m!!(2l-1)!!}{(m+1)!!\sqrt{(2l)!}} A\pi^{1/4}; \qquad a_{2l+1} = \pm \frac{3}{2\sqrt{2}} \frac{(m-2)!!(2l+1)!!}{(m+1)!!\sqrt{(2l+1)!}} \frac{B}{A} \pi^{1/4};$$

$$L_{2m} = \frac{m!!}{(m+1)!!}\sqrt{\pi}\left(A^2 + \frac{3B^2}{8mA^2}\right). \tag{7}$$

(Here, for definiteness, m is assumed to be even.) Particular cases of formulas (7) are the formulas obtained by I. A. Grigorieva [(3)].

We note that the theorems of the present note may fail if the number of conditions (2) is more than one for $y_n(x) \in T_n^{(h)}$ and more than two for $y_n(x) \in B_n^{(h)}$ or for $f_{2m}(x) \in G_{2m}$, as was also noted in note [(1)], since for $h = 0$ the classes $T_n^{(h)}$ and $B_n^{(h)}$ reduce to the classes T_n and B_n considered in [(1)].

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CITED LITERATURE

[1] B. A. Rymarenko, DAN, **103**, No. 3 (1955).
[2] B. A. Rymarenko, Dokl. AN USSR, No. 2 (1952).
[3] I. A. Grigorieva, Izv. Kiev Polytechnic Institute, **6** (1954).

Note: Figure translations are in progress. See original paper for figures.

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