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Abstract

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SOLUTION OF CERTAIN BINARY ADDITIVE PROBLEMS BY COMPUTING THE VARIANCE IN PROGRESSIONS

In my preceding note ([1]) the concept of the variance of quadratic forms, including the simplest forms $\varphi = \xi^2 + \eta^2$ and $\varphi = \xi\eta$, in arithmetic progressions was considered, and it was explained how, by means of this concept, one may solve certain binary problems. A further improvement of this method has made it possible to solve binary problems of the form:

$$n = uv + Q(\xi, \eta), \tag{1}$$

where u and v independently run through quite arbitrary sequences of numbers, and $Q(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$ is a binary quadratic form. In this case the corresponding asymptotic laws are obtained.

As an example of the theorems obtained, let us consider the equation

$$n = p_1 p_2 + \xi^2 + \eta^2, \tag{2}$$

where p_1 and p_2 are prime numbers.

Theorem. Let $\alpha > 0$ be a sufficiently small number; $N_1 = n^{1-\alpha}$, $N_2 = n^{\alpha}$; p_1 runs through primes $\leq N_1$; p_2 runs through primes $\leq N_2$; $\xi^2 + \eta^2$ runs through the integers free of squares. Then the number $H(n)$ of solutions of (2) has the form

$$H(n)\,\mathrm{Li}(N_1)\,\mathrm{Li}(N_2)\,\pi \prod_p \left(1 + \frac{\chi_4(p)}{p(p-1)}\right) \prod_{p>2} \left(1 - \frac{2 + \chi_4(p) - \frac{1}{p} - \frac{\chi_4(p)}{p}}{p^2 - p + \chi_4(p)}\right) \times \prod_{p \mid n,\, p>2} \frac{(p - \chi_4(p))(p-1)}{p^2 - p - 2 + \frac{1}{p} + \frac{\chi_4(p)}{p}} + O\left(\frac{N_1 N_2}{(\ln n)^C}\right), \tag{3}$$

where C is an arbitrarily large constant.

Let us briefly explain the ideas of the proof of this theorem. Let

$$U(m) = \sum_{m=\xi^2+\eta^2} 1; \qquad D_1 = N_1 = n^{1-\alpha}; \quad D_2 = n^{1-\alpha-\zeta_0}$$

(in what follows ζ_i, α_i are small positive constants); D runs through the numbers $D_1 - D_2 \leq D \leq D_1$; v runs through the prime numbers of the segment $[1, N_2]$ $(N_2 = n^\alpha)$.

Let (QFR) be the sequence of integers free of squares. For brevity of exposition (and only for this purpose) we shall henceforth suppose u to be a prime number.

$$r(n, D) = \sum_{\substack{m \leq n \\ m=n-Dv\in(QFR) \\ v \text{ prime}}} U(m) - A(n, D), \tag{4}$$

$$A(n, D) = \mathrm{Li}(N_2)\, \pi \prod_p \left(1 + \frac{\chi_4(p)}{p(p-1)}\right) \prod_{p>2} \left(1 - \frac{2 + \chi_4(p) - \frac{1}{p} - \frac{\chi_4(p)}{p}}{p^2 - p + \chi_4(p)}\right) \times$$

$$\times \prod_{\substack{p|D \\ p>2}}{}^{*} \frac{(p - \chi_4(p))(p-1)}{p^2 - p - 2 + \frac{1}{p} + \frac{\chi_4(p)}{p}}. \tag{5}$$

In the present case, the dispersion in progressions is the expression

$$V = \sum_D (r(n, D))^2 = \sum_D \left(\sum_{\substack{m \leq n \\ m=n-D\nu\in(QFR) \\ \nu \text{ prime}}} U(m) - A(n, D) \right)^2. \tag{6}$$

Main lemma.

$$V \ll D_2 N_2^2 \frac{1}{(\ln n)^C}; \tag{7}$$

here C is an arbitrarily large constant.

The solution of problem (2) with the aid of the main lemma is quite simple: if D runs only through prime numbers, then, by discarding the remaining numbers, we only decrease the dispersion, so that

$$\sum_{D \text{ prime}} \left(\sum_{\substack{m \le n \\ m = n - D\nu \in (QFR) \\ \nu \text{ prime}}} U(m) - A(n, D) \right)^2 \ll D_2 N_2^2 \frac{1}{(\ln n)^C}. \qquad (8)$$

It follows at once from this that there exist representations $n = D\nu + \xi^2 + \eta^2$, where D and ν are prime and $D_1 - D_2 \le D < D_1$. Indeed, if there were no such solutions, then the terms $\sum U(m)$ in the parentheses in (8) would vanish, and the resulting sum would coincide with

$$\sum_{D \text{ prime}} (A(n, D))^2 > \frac{N_2^2 D_2}{\ln^4 n},$$

which would contradict (8). A simple computation also gives the asymptotic formula (9).

Let us explain the principles of evaluating (6). The parentheses on the left-hand side of (6) are expanded, and the order of summation is changed: first we sum over D. The main difficulty is the summation of the term

$$\sum_D \left(\sum_\nu U(n - D\nu) \right)^2 = \sum_{\nu_1, \nu_2, D} U(n - D\nu_1) U(n - D\nu_2).$$

Here ν_1, ν_2 run through identical sequences of prime numbers; $n - D\nu_i \in (QFR)$ $(i = 1, 2)$.

Further, from elementary considerations we derive, for given ν_1, ν_2,

$$\sum_D U(n - D\nu_1) U(n - D\nu_2) =$$

$$= \frac{1}{4} \sum_{\substack{\delta \in (QFR) \\ \delta \le n}} (U(\delta))^2 \sum_{\substack{t \in (QFR) \\ (t, \delta) = 1 \\ t \le n}} \mu(t) \sum_{\substack{D \\ n - D\nu_i \in (QFR) \\ t\delta | n - D\nu_i \ (i = 1, 2)}} U\left(\frac{n - D\nu_1}{\delta} \frac{n - D\nu_2}{\delta} \right). \qquad (9)$$

Thus the matter is reduced to counting, as D varies, the number of representations

$$(n - D\nu_1)(n - D\nu_2) = \delta^2 (\xi^2 + \eta^2)$$

or

$$n^2(\nu_1 - \nu_2)^2 = (4\nu_1\nu_2 D - n(\nu_1 + \nu_2))^2 - 4\delta^2\nu_1\nu_2(\xi^2 + \eta^2), \qquad (10)$$

* For simplicity it is assumed here that n has no small divisors.

i.e., the square number $n^2(\nu_1-\nu_2)^2$ must be represented by the ternary quadratic form $x^2 - 4\delta^2\nu_1\nu_2(\xi^2 + \eta^2)$ under certain geometric and congruence conditions on x. Such a problem is solved quite satisfactorily (although rather cumbersomely) in the more general theory of the author and A. V. Malyshev (see the literature in note ([1])). The only computational difficulty is the calculation of the corresponding arithmetical factor.

The general problem (1) is treated analogously. The method can also be applied to decompositions of the form

$$n = N(\mathfrak{a}) + Q(\xi, \eta), \qquad (11)$$

where $N(\mathfrak{a})$ is the norm of an ideal from any (not only Abelian, as in note ([1])) field, and moreover the ideal $\mathfrak{a}$ may be taken from any prescribed class. An asymptotic law also holds. This generalizes Kloosterman' s theorem on an equation of the form $n = ax^2 + by^2 + cz^2 + dt^2$.

Finally, the general solution of equation (1) can be applied to the study of the well-known Hardy-Littlewood equation ([2])

$$n = p + \xi^2 + \eta^2. \qquad (12)$$

This, however, requires very laborious computations.

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CITED LITERATURE

[1] Yu. V. Linnik, DAN, **120**, No. 5, 960 (1958).
[2] H. H. Hardy, J. E. Littlewood, Acta Math., **44** (1923).

Note: Figure translations are in progress. See original paper for figures.