MATHEMATICS

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1958

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Abstract

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DEFINING RELATIONS IN CATEGORIES

The immediate purpose of the present note is to transfer the theory of defining relations to classes of models. However, the basic structural concepts of model theory—submodels, direct products, and some others—can be expressed through the concept of homomorphism in the sense of ([1]), and moreover in the same way as in the general theory of categories. Therefore below defining relations are introduced and studied at once in general categories. At the end their interpretation for categories of models is indicated.

1. According to Eilenberg and Mac Lane ([2]), a **category** K is a class of elements in which a partial binary operation of multiplication and a predicate of being a neutral element are defined, subject to the axioms: 1) if ab, bc are defined, then $a \cdot bc$ and $ab \cdot c$ are defined and equal; 2) if $a \cdot bc$ or $ab \cdot c$ is defined, then ab is defined; 3) if e is a neutral element, then $e^2 = e$, and from the definedness of ea or ae there follows, respectively, $ea = a$ or $ae = a$; 4) for each $a \in K$ there exist neutral elements e, e' such that $ea = ae' = a$. The neutral elements of K are also called **objects** of the category K. If for objects e, e' we have $ea = ae' = a$, then a is called a **homomorphism** from e to e'. A homomorphism a is called an **isomorphism** if there exists in K a homomorphism b for which ab and ba are neutral elements of K. In this case the objects ab, ba are called isomorphic.

According to Mac Lane ([3]), an object $\mathfrak{A}$ of a category K is called a **direct composition** of a system of objects $\mathfrak{A}_\alpha$ $(\alpha \in \Gamma)$, if there exist homomorphisms π_α of the object $\mathfrak{A}$ into $\mathfrak{A}_\alpha$ such that for any system of homomorphisms σ_α of an arbitrary object $\mathfrak{B}$ into $\mathfrak{A}_\alpha$ there is one and only one homomorphism $\xi : \mathfrak{B} \to \mathfrak{A}$ satisfying the relations $\sigma_\alpha = \xi\pi_\alpha$ $(\alpha \in \Gamma)$. Dually to this, an object $\mathfrak{A} \in K$ is called a **K-free composition** of the system $\mathfrak{A}_\alpha$ $(\alpha \in K)$, if there exist homomorphisms $\pi_\alpha : \mathfrak{A}_\alpha \to \mathfrak{A}$ such that for any system of homomorphisms σ_α of the objects $\mathfrak{A}_\alpha$ into an arbitrary K-object $\mathfrak{B}$ there is one and only one homomorphism $\xi : \mathfrak{A} \to \mathfrak{B}$ satisfying the relations $\sigma_\alpha = \pi_\alpha\xi$ $(\alpha \in \Gamma)$. Direct and free compositions may fail to exist, but if they exist, they are determined uniquely up to isomorphism.

Further, a **subcategory** L of a category K will mean a subclass of the class of objects of K, together with all homomorphisms from K belonging to all possible pairs of L-objects. The meaning of the concepts of isomorphism, direct and free composition in general changes when passing from K to L. However, isomorphism and direct composition will henceforth be used in the sense of the basic category, while free composition will be considered in various subcategories in the sense of these subcategories.

2. We shall next need categories of a more special kind, which will be called categories of structured sets, or simply categories of structures. Analogous concepts have been considered by Isbell ([4]) and by Mac Lane himself ([3]). At the foundation is placed a class K of objects, called structures (structured sets, spaces). With each-

with each structure $\mathfrak{A}$ there is assumed to be associated a uniquely determined set $\mu(\mathfrak{A})$—the **underlying set** of the structure $\mathfrak{A}$. In addition, there is specified a certain system H of single-valued mappings of underlying sets of structures into underlying sets of structures, and it is required that the identity mapping of each structure onto itself belong to H, and that for a mapping α of a structure $\mathfrak{A}$ into a structure $\mathfrak{B}$ and a mapping β of a structure $\mathfrak{B}$ into a structure $\mathfrak{C}$, from $\alpha, \beta \in H$ it follow that $\alpha\beta \in H$. The mappings from H are called **homomorphisms**. We shall regard as **identical** structures with a common underlying set whose identity mapping onto itself is a homomorphism of the first structure into the second and of the second into the first. We shall also assume that a structure is defined on an arbitrary set A if a one-to-one mapping of A onto the underlying set of some structure $\mathfrak{A}$ is given and it is said that this mapping is an isomorphism. A class of structures is **abstract** if, together with each structure, it contains all structures isomorphic to it. Below only abstract classes of structures are considered. The totality H of all homomorphisms of K-structures is a category in the sense described. The objects of this category will be identified with the structures whose identity mappings they are.

In a category of structures K, a structure $\mathfrak{B}$ is called a **substructure** of a structure $\mathfrak{A}$ if: 1) $\mu(\mathfrak{B}) \subseteq \mu(\mathfrak{A})$; 2) every homomorphism of $\mathfrak{A}$ into an arbitrary structure $\mathfrak{C}$, considered for $\mathfrak{B}$, is a homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$; 3) every mapping of a structure $\mathfrak{C}$ into $\mathfrak{B}$ that is a homomorphism of $\mathfrak{C}$ into $\mathfrak{A}$ is a homomorphism of $\mathfrak{C}$ into $\mathfrak{B}$. The substructure $\mathfrak{B}$ is uniquely determined by $\mu(\mathfrak{B})$ and $\mathfrak{A}$. A structure $\mathfrak{B}$ will be called a **strong** substructure in $\mathfrak{A}$ if, in addition to conditions 1-3, we have: 4) every homomorphism into $\mathfrak{B}$ is a homomorphism into $\mathfrak{A}$.

The direct composition $\mathfrak{A}$ of a system of structures $\mathfrak{A}_\alpha$ will be called **separating** if from "$a\pi_\alpha = b\pi_\alpha$ for all canonical homomorphisms π_α $(\alpha \in \Gamma)$" it follows that $a = b$ $(a, b \in \mathfrak{A})$. This composition will be called **complete** if for any $a_\alpha \in \mathfrak{A}_\alpha$ in $\mathfrak{A}$ there is an element a such that $a_\alpha = a\pi_\alpha$ $(\alpha \in \Gamma)$. A direct composition is called a **direct product** if it is separating and complete. In this case the elements of the direct composition will be identified with the elements of the Cartesian product of the underlying sets of the factors.

A structure $\mathfrak{A}$ of a category K will be called **unitary** if $\mathfrak{A}$ is one-element and the mapping of any K-structure into $\mathfrak{A}$ is a homomorphism. $\mathfrak{A}$ will be called a **zero-structure** if $\mathfrak{A}$ is one-element and every mapping of $\mathfrak{A}$ into an arbitrary K-structure is a homomorphism. It is easy to see that if a category K contains a zero-structure, then every direct composition will be complete and separating. It is just as easy to prove:

Theorem 1. *Suppose that in the category K all substructures are strong and direct compositions are separating, and suppose that in each K-structure $\mathfrak{A}_\alpha$ $(\alpha \in \Gamma)$ a K-substructure $\mathfrak{B}_\alpha$ is distinguished and that the direct compositions $\mathfrak{A} = \prod \mathfrak{A}_\alpha$, $\mathfrak{B} = \prod \mathfrak{B}_\alpha$ exist. Then $\mathfrak{B}$ is a substructure of $\mathfrak{A}$.*

We shall agree to call a category L **multiplicatively closed** if direct compositions of arbitrary systems of L-structures exist in it.

Theorem 2. *In order that, in a multiplicatively closed category K containing a unitary structure, with strong substructures and separating direct compositions, all canonical homomorphisms of K-structures into their K-free compositions be isomorphisms onto the corresponding substructures, it is necessary and sufficient that every K-structure be isomorphically embeddable in a K-structure with a unitary substructure.*

In the consideration of categories of structures below, the following **axiom of definiteness** will always be assumed to hold: the totality of all K-structures defined on any fixed set,

can be regarded as a set of a certain cardinality. We shall call a category of structures K **bounded** if, for every cardinal number $\mathfrak{m}$, there exists a cardinal $\mathfrak{n}+K)(\mathfrak{m})$ such that in every K-structure any set of elements of cardinality $\leq \mathfrak{m}$ is contained in a suitable K-substructure of cardinality $\leq \mathfrak{n}$.

3. Let K_0 be some (general) category K, and let L be its subcategory, with $K \supset L$. A **replica** of a K-object $\mathfrak{A}$ in the category L (an L-replica of $\mathfrak{A}$) is an L-object $\mathfrak{B}$ for which there exists a homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that, for every homomorphism σ of the object $\mathfrak{A}$ into an arbitrary L-object $\mathfrak{C}$, there is one and only one homomorphism $\xi : \mathfrak{B} \to \mathfrak{C}$ satisfying the relation $\sigma = \pi\xi$. The homomorphisms π, ξ will be called canonical. It is easy to see that, if a replica exists, then it is determined up to isomorphism by the object $\mathfrak{A}$, and, in particular, the L-replica of an L-object always coincides with the object itself.

Theorem 3. *Let $K \subset L \subset M$ be subcategories, and let $\mathfrak{A}^L, \mathfrak{A}^M$ be the replicas of a K-object $\mathfrak{A}$ in L and M. Then $\mathfrak{A}^M$ is the M-replica of the object $\mathfrak{A}^L$. If $\mathfrak{A}$ is the L-free composition of K-objects $\mathfrak{A}_\alpha(\alpha \in \Gamma)$ and the L-replicas $\mathfrak{A}^L, \mathfrak{A}^L_\alpha$ of these objects exist, then $\mathfrak{A}^L$ is the L-free composition of the system $\mathfrak{A}^L_\alpha$.*

Let us note that if categories of structures are considered and $\mathfrak{B}$ is an L-replica of the structure $\mathfrak{A}$ with canonical homomorphism π, then in $\mathfrak{B}$ there is no L-substructure distinct from $\mathfrak{B}$ and containing $\mathfrak{A}^\pi$, i.e. $\mathfrak{B}$ is generated by the elements $\mathfrak{A}^\pi$.

A set of elements S of a structure $\mathfrak{A}$ of the category K will be called **K-dense** in $\mathfrak{A}$ if, for any homomorphisms η, σ of the structure $\mathfrak{A}$ into an arbitrary K-structure, from "$a\eta = a\sigma$ for all $a \in S$" it follows that $\eta = \sigma$.

It is easy to see that if π_α are the canonical homomorphisms of the free composition $\mathfrak{A}$ of structures $\mathfrak{A}_\alpha$ ($\alpha \in \Gamma$), then the set $\bigcup \mathfrak{A}_\alpha^{\pi_\alpha}$ will be K-dense in $\mathfrak{A}$. Further, if in $\mathfrak{A}$ there is a K-substructure $\mathfrak{B}$ containing the set $\bigcup \mathfrak{A}_\beta^{\pi_\beta}$ ($\beta \in \Gamma_1 \subset \Gamma$) as its dense system, then $\mathfrak{B}$ will be the K-free composition of the structures $\mathfrak{A}_\beta$ ($\beta \in \Gamma_1$). It also follows from the definition that the canonical image of a structure in its replica is a dense set in the latter.

We shall call a subcategory L in a category K **R-complete** if every K-object has a replica in L. From Theorem 3 it is clear that an R-complete subcategory of an R-complete subcategory will be R-complete.

Theorem 4. *If the category K is multiplicatively closed, then every R-complete subcategory of it is multiplicatively closed. If K contains a unit structure, then every R-complete subcategory of the category K contains it as well.*

A structure $\mathfrak{A}$ of a category K will be called **regular** if, for every set S of elements of $\mathfrak{A}$, there exists a substructure containing S as its K-dense set. The category K will be called **regular** if all its structures are regular.

Theorem 5. *If a subcategory L of a category of structures K contains a unit structure, is multiplicatively closed, regular, and bounded, then L is R-complete and every system of L-structures has a definite L-free composition.*

The proof is analogous to the proof of the theorem on the existence of topological algebras with given defining relations and a topological generating space ([5]).

4. Denote by K the class of all models of a fixed type $\langle P_1, P_2, ...\rangle$, where P_i are the basic predicate symbols of the class.

According to ([1]), a single-valued mapping σ of the model $\langle A;\ P_1, P_2, ...\rangle$ into the model $\langle B;\ Q_1, Q_2, ...\rangle$ is called a homomorphism if, for any i and $a_1, a_2, ...$ from A, from the truth of $P_i(a_1, a_2, ...)$ follows the truth of $Q_i(a_1\sigma, a_2\sigma, ...)$.

The totality H of all homomorphisms of K-models thus defined satisfies the requirements of item 1, and the pair (K, H) is a category of structures–models of the given type. It is easily verified that here the notions of submodel and direct product of models coincide with the notions of substructure and direct composition; the unit structure is the one-element model on which all basic predicates are true, and the zero structure is the one-element model with false basic predicates.

Let now there be given some subclass L of the class of models K, a system A of symbols, and a system F of formulas of the form $P_i(a_1, a_2, ...)$, $a_\nu \in A$. An L-model $\mathfrak{B}$ is naturally called a model with generators A and defining relations F in the class L, if there exists a mapping π of the system A into $\mathfrak{B}$ having the following properties: a) A^π is L-dense in $\mathfrak{B}$; b) the expressions $P_i(a_1^\pi, a_2^\pi, ...)$ are true in $\mathfrak{B}$; c) for every mapping σ of A into an arbitrary L-model $\mathfrak{C}$ such that $P_i(a_1^\sigma, a_2^\sigma, ...)$ are true in $\mathfrak{C}$, there exists a homomorphism $\xi : \mathfrak{B} \to \mathfrak{C}$ satisfying the relation $\sigma = \pi\xi$.

Denoting by $\mathfrak{A}$ the model with basic set A on which the expressions from F are true and the remaining expressions of similar form are false, we see that the model $\mathfrak{B}$ with generators A and defining relations F will be isomorphic to the L-replica of $\mathfrak{A}$ in L.

One may also take as the basic class K the totality of all possible models of the given type on whose basic sets certain topologies are defined, and take as H the totality of continuous mappings which are simultaneously homomorphisms in the sense of the theory of models. If now by L one understands the totality of topological algebras of the corresponding type, then the conditions of Theorem 5 are fulfilled, and the L-replica of the K-structure $\mathfrak{A}$ will be the topological algebra given, in the sense of (5), by the topological space $\mathfrak{A}$ and by the positive description of $\mathfrak{A}$ in the above sense.

Received
29 I 1958

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Note: Figure translations are in progress. See original paper for figures.