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Abstract

Full Text

MATHEMATICS

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SOME QUESTIONS ON THE APPROXIMATION OF FUNCTIONS OF ONE VARIABLE BY ALGEBRAIC POLYNOMIALS

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Using the inequalities that were formulated in note (1), and the notation introduced there, we prove a number of direct and inverse theorems in the theory of approximation of continuous functions by algebraic polynomials in the metric $L_p(-1,+1)$, $1 \leqslant p \leqslant \infty$.

Theorem 1. Let m be a natural number $\geqslant 1$, $1 \leqslant p \leqslant \infty$, $1+s-\frac{1}{p} \geqslant 0$, and let the function $\omega(t)$ satisfy condition A_α^β, where $\alpha = 0$, $0 \leqslant \beta^*$. Then there exists a constant $A = A(m, s, \beta)$ such that, whatever the function $f(x)$, defined on $[-1,+1]$, having an $(m-1)$-st absolutely continuous derivative and a derivative $f^{(m)}(x)$ of order m integrable in the p-th degree, and whatever the algebraic polynomial $P_n(x)$, there exist polynomials $P_{n+i}(x)$ $(i = 1, 2, \dots, m)$ such that the inequality holds

$$\left\| \frac{f^{(m-i)}(x) - P_{n+i}(x)}{g^{i+s}(x,n)\,\omega[g(x,n)]} \right\|_{L_p(-1,+1)} \leqslant A \left\| \frac{f^{(m)}(x) - P_n(x)}{g^{s}(x,n)\,\omega[g(x,n)]} \right\|_{L_p(-1,+1)}$$

$$\left(g(x,t) = \frac{\sqrt{1-x^2}}{t} + \frac{1}{t^2}; \quad n = 1, 2, \dots \right).$$

Theorem 2. If $f(x)$ has on the interval $[-1,+1]$ an m-th continuous derivative $f^{(m)}(x)$, satisfying on $[-1,+1]$ the condition of quasismoothness

$$\left| f^{(m)}(x_1) + f^{(m)}(x_2) - 2f^{(m)}\left(\frac{x_1+x_2}{2}\right) \right| \leqslant M|x_1 - x_2|,$$

then there exists a sequence of algebraic polynomials $P_n(x)$ such that

$$|f(x) - P_n(x)| \leqslant C(m) M g^m(x,n) \left[g(x,n) + \frac{\ln n}{n^2} \right] \quad (n = 1, 2, \dots).$$

Since a function $f(x)$ having a derivative bounded on $[-1,+1]$ satisfies the condition of quasismoothness, Theorem 2 is, for $m=0$, a generalization of the corresponding result of S. M. Nikol′skii (2), though with a cruder constant $C(0)$.

* In particular, this condition is satisfied for $\omega(t)=t^{\beta}$ when $\beta \geqslant 0$.

Theorem 3. If the m-th derivative of the function $f(x)$, defined on the interval $[-1,+1]$, has the property that for it

$$\left\|(\sqrt{1-x^2})^s f^{(m)}(x)\right\|_{L_p(-1,+1)} \leq M \qquad \left(1-s-\frac{1}{p}\geq 0,\ p\geq 1\right),$$

then there exists a sequence of algebraic polynomials $P_n(x)$ such that

$$\left\|\frac{f(x)-P_n(x)}{g^{m-s}(x,n)}\right\|_{L_p(-1,+1)} \leq M\,\frac{C(m)}{n^s} \qquad (n=1,2,...). \tag{1}$$

For $m=1,\ s\leq 0$, this theorem follows from a result of M. K. Potapov ([3]), but for $s\geq 0$, conversely, M. K. Potapov' s result follows from Theorem 3.

Definition. We shall say that $f(x)$, defined on the interval $[-1,+1]$, belongs to the class $H_\omega^{(r)}(1)L_p$, respectively to the class $H_\omega^{(r)}(2)L_p$ (r is an integer $\geq 0,\ 1\leq p\leq\infty$), if the following conditions are satisfied:

$$\left\|\frac{f^{(r)}\left(x\sqrt{1-h^2}-h\sqrt{1-x^2}\right)-f^{(r)}(x)}{\omega\left[|h|\sqrt{1-x^2+h^2}\right]}\right\|_{L_p(-1,+1)} \leq M, \tag{*}$$

respectively

$$\left\|\frac{f^{(r)}\left(x\sqrt{1-h^2}-h\sqrt{1-x^2}\right)+f^{(r)}\left(x\sqrt{1-h^2}+h\sqrt{1-x^2}\right)-2f^{(r)}(x)}{\omega\left[|h|\sqrt{1-x^2+h^2}\right]}\right\|_{L_p(-1,+1)} \leq M$$

$$(0<h\leq 1) \tag{**}$$

for some function $\omega(t)$ satisfying the conditions of Theorem 1.

Theorem 4. If $f(x)\in H_\omega^{(m)}(1)L_p$ or $f(x)\in H_\omega^{(m)}(2)L_p$, then there exists a sequence of algebraic polynomials $P_n(x)$ for which

$$\left\|\frac{f(x)-P_n(x)}{g^m(x,n)\,\omega[g(x,n)]}\right\|_{L_p(-1,+1)} \leq C(m)M \qquad (n=1,2,...). \tag{2}$$

Let us make the corresponding remark. Suppose on the interval $[-1,+1]$ there is given a function $f(x)$ having on it a continuous m-th derivative $f^{(m)}(x)$, satisfying the inequality (for $C=1$)

$$|f^{(m)}(x_1)-f^{(m)}(x_2)| \le C\,\omega(|x_1-x_2|) \qquad (x_1,x_2\in[-1,+1]), \tag{3}$$

where

$$\omega(t)=\sup_{|x_1-x_2|\le t}|f^{(m)}(x_1)-f^{(m)}(x_2)|.$$

It follows easily from this that the inequality

$$\left|\frac{f^{(m)}\left(x\sqrt{1-h^2}-h\sqrt{1-x^2}\right)-f^{(m)}(x)}{\omega\left[|h|\sqrt{1-x^2+h^2}\right]}\right|\le 1 \qquad (0<|h|\le 1,\ |x|\le 1),$$

holds; consequently, on the basis of Theorem 4, for $p=\infty$ there exists a sequence of polynomials $P_n(x)$ such that

$$|f(x)-P_n(x)|\le C(m)g^m(x,n)\omega[g(x,n)], \qquad (n=1,2,...); \tag{4}$$

this inequality was first obtained by A. F. Timan ([4]).

Let $r\ge 0,\ r=\bar{r}+\alpha,\ \bar{r}$ an integer $\ge 0,\ 0<\alpha\le 1,\ 1\le p\le\infty$. By definition (see ([5]), pp. 261-326), a function $f(x)\in H_p^{(r)}(-1,+1;M)$ if it and its derivatives up to order $\bar{r}$ inclusive belong to

$L_p(-1,+1)$, and the inequalities hold

$$\left(\int_{-1}^{1-h}\left|\overline{f^{(r)}}(x+h)-\overline{f^{(r)}}(x)\right|^p dx\right)^{1/p}\le Mh^\alpha \qquad (0<\alpha<1;\ 0<h<2);$$

$$\left(\int_{-1+h}^{1-h}\left|\overline{f^{(r)}}(x+h)+\overline{f^{(r)}}(x-h)-2\overline{f^{(r)}}(x)\right|^p dx\right)^{1/p}\le Mh^\alpha \qquad (\alpha=1,\ 0<h<1).$$

We shall agree below that a function $f(x)\in\overline{H}_p^{(r)}=\overline{H}_p^{(r)}(-1,+1;M)$ if it, together with its derivatives up to order r inclusive, belongs to $L_p(-1,+1)$, and if, for it, for $0<\alpha<1$ the inequality $(*)$ holds, where $\omega(t)=t^\alpha$, and for $\alpha=1$ the inequality $(**)$ holds, where $\omega(t)=t$. There is a close connection between the classes $H_p^{(r)}$ and $\overline{H}_p^{(r)}$ (see below). At the same time, when considering one and the same function in L_p and L_∞, the functions of the indicated classes exhibit properties analogous to those known properties (see ([5]), pp. 261-326) that hold for functions of the classes $H_p^{(r)}$ defined on the entire real axis. However, despite the presence of a close connection between functions of the classes $H_p^{(r)}$ and $\overline{H}_p^{(r)}$, the problem of approximating functions $f(x) \in H_p^{(r)}$ by algebraic polynomials is still not completely solved. We note that from Theorem 1 and from the fact that for $f(x) \in H_p^{(r+\alpha)}$ there exists a sequence of algebraic polynomials such that

$$\|f(x) - P_n(x)\|_{L_p(-1,+1)} \leq M \frac{C(\overline{r})}{n^{\overline{r}+\alpha}},$$

one easily obtains the result of M. K. Potapov ([3]).

We now turn to inverse theorems in the theory of approximation of functions. Let $\omega(t)$ be continuous on the interval $[-1,+1]$ and satisfy the conditions:

1) $\omega(t) \geq 0$, if $t \geq 0$;
2) $\omega(t)$ is nondecreasing on the interval $[0,2]$;
3) $$\sum_{k=0}^{N} \frac{\omega\left[g(x,a^k)\right]}{g(x,a^k)} \leq C\, \frac{\omega\left[g(x,a^N)\right]}{g(x,a^N)};$$
4) $$\sum_{k=N+1}^{\infty} \omega\left[g(x,a^k)\right] \leq C\omega\left[g(x,a^{N+1})\right]$$

   (C is a certain constant independent of x and N) for some natural number $a > 1$.

We shall say that $\omega(t)$ satisfies condition (A) if 1), 2), 3), 4) are fulfilled, and condition (B) if 1), 2), 4) are fulfilled.

Theorem 5. *Let $f(x)$ be defined on $[-1,+1]$, and suppose that there exists for it a sequence of algebraic polynomials $P_n(x)$ such that inequality (4) is satisfied. Suppose, moreover, that $\omega(t)$ satisfies condition (A) or (B). Then the function $f(x)$ has on $[-1,+1]$ a continuous derivative $f^{(m)}(x)$ of order m, satisfying, in case (A), inequality (3) with some constant C, and, in case (B), the inequalities*

$$\left|f^{(m)}(x_1) - f^{(m)}(x_2)\right| \leq C\omega(|x_1 - x_2|) \ln \frac{2}{|x_1 - x_2|},$$

$$\left|f^{(m)}(x_1) + f^{(m)}(x_2) - 2f^{(m)}\left(\frac{x_1+x_2}{2}\right)\right| \leq C\omega(|x_1 - x_2|) \qquad (x_1, x_2 \in [-1,+1]).$$

We note that the function $\omega(t) = t^\alpha$ for $0 < \alpha < 1$ satisfies condition (A), and for $\alpha = 1$ condition (B). In this case we obtain the theorems of V. K. Dzyadyk ([6])*.

* This theorem was obtained by us by another method, independently of V. K. Dzyadyk, and was reported at the seminar on the theory of approximation of functions at the V. A. Steklov Mathematical Institute of the Academy of Sciences of the USSR at the end of 1956.

Theorem 6. Let $r + \alpha - \dfrac{1}{p} > 0$ (r an integer ≥ 0, $0 < \alpha \leq 1$), and let, for a function $f(x) \in L_p(-1, +1)$, defined on $[-1, +1]$, there exist a sequence of algebraic polynomials $P_n(x)$ such that

$$\left\| \frac{f(x) - P_n(x)}{g^{r+\alpha}(x, n)} \right\|_{L_p(-1,+1)} \leq M \qquad (n = 1, 2, ...). \tag{5}$$

Then $f(x) \in H_\infty^{(r+\alpha-1/p)}(-1, +1, CM)$, where C is some constant.

Theorem 7. Let $\alpha - \dfrac{1}{p} > 0$, and let, for a function $f(x) \in L_p(-1, +1)$, there exist a sequence of algebraic polynomials $P_n(x)$ $(n = 1, 2, ...)$ for which inequality (5) holds. Then $f(x) \in \overline{H}_p^{(r+\alpha)}(-1, +1; CM)$, where C is some constant.

Theorem 8. Let $m > 0$ be an integer, and suppose that $f(x)$ has the property that there exists for it a sequence of polynomials $P_n(x)$ such that

$$\left\| \frac{f(x) - P_n(x)}{g^{m-s}(x, n)} \right\|_{L_p(-1,+1)} \leq \frac{M}{n^s} \qquad (n = 1, 2, ...).$$

Then:

1) if $s \leq 0$, then $f(x) \in H_\infty^{(m-1/p)}(-1, +1; CM)$;

2) if $s > 0$, $m - \dfrac{s}{2} - \dfrac{1}{p} > 0$, then
$f(x) \in H_\infty^{(m-s/2-1/p)}(-1, +1, CM)$,

where C is some constant.

Let us note some properties of functions of the classes under consideration.

1) If $f(x)$ satisfies condition (3), then $f(x) \in H_\omega^{(m)}(1)L_\infty$. The converse inclusion is true for $\omega(t)$ satisfying condition (A).

2) If $f(x) \in \overline{H}_\infty^{(r)}$ (r an integer ≥ 1), then $f(x) \in H_\infty^{(r)}(-1, +1, CM)$. The question of the converse inclusion remains open for the time being.

3) If $f(x) \in H_\omega^{(r)}(2)L_p$ ($\omega(t) = t^\alpha$, $0 < \alpha < 1$), then $f(x) \in H_\omega^{(r)}(1)L_p$.

4) If $f(x) \in \overline{H}_p^{(r)}(-1, +1; CM)$, $f(x) \in H_\infty^{(r_1)}(-1, +1; CM)$, where

$$r_1 = r + \alpha - \frac{1}{p} \qquad \left(r + \alpha - \frac{1}{p} > 0\right).$$

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Note: Figure translations are in progress. See original paper for figures.