HYDROMECHANICS

1958

SovietRxiv

Abstract

Full Text

HYDROMECHANICS

A. G. KULIKOVSKII

ON MOTIONS WITH HOMOGENEOUS DEFORMATION IN MAGNETOHYDRODYNAMICS

(Presented by Academician L. I. Sedov, 23 I 1958)

Below we indicate a class of solutions of the equations of magnetohydrodynamics (for infinite electrical conductivity of the medium) for which the law of motion in Lagrangian form has the form

$$x_i = M_{ij}(t)x_j^0 + M_i(t), \tag{1}$$

where x_i are the orthogonal coordinates of a particle; x_i^0 are the same coordinates at $t = 0$. Similar motions were considered earlier in the works ($^{1-3}$). The equations of continuity, adiabaticity, and frozen-in magnetic lines of force are then written in the form

$$\rho = \frac{1}{\Delta}\rho^0, \qquad p = \frac{1}{\Delta^\gamma}p^0, \qquad H_i = \frac{1}{\Delta}M_{ij}H_j^0. \tag{2}$$

Here Δ is the determinant of the matrix $\|M_{ij}\| = M$.

Substituting these equalities into the equations of motion

$$\frac{d^2x}{dt^2} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial T_{ij}}{\partial x_j}, \qquad \text{where} \quad T_{ij} = \frac{1}{4\pi}H_{iH}j - \frac{1}{8\pi}H_{kH}k,$$

and taking account of equalities (1), we obtain

$$\frac{d^2M_i}{dt^2} + x_j^0\frac{d^2M_{ij}}{dt^2} = -\frac{1}{\rho^0\Delta^{\gamma-1}}\frac{\partial p^0}{\partial x_q^0}N_{qi} +$$

$$+\frac{1}{4\pi\rho^0\Delta}\left[\frac{\partial H_k^0H_l^0}{\partial x_l^0}M_{ik} - \frac{1}{2}\frac{\partial H_n^0H_m^0}{\partial x_h^0}M_{sn}M_{sm}N_{hi}\right] \quad (\|N_{ij}\| = M^{-1}). \tag{3}$$

Represent the matrix M as a product

$$M = A \cdot B,$$

where A is an orthogonal matrix and B is symmetric. Then the right-hand side of equality (3) can be written in the following form:

$$\frac{1}{\rho^0} A_{ij} \left\{ -\frac{1}{\Delta^{\gamma+1}} D_{jq} \frac{\partial p^0}{\partial x_q^0} + \frac{1}{4\pi\Delta} \left[B_{jk} \frac{\partial H_k^0 H_l^0}{\partial x_l^0} - \frac{1}{2} D_{jh} B_{sn} B_{sm} \frac{\partial H_n^0 H_m^0}{\partial x_h^0} \right] \right\}, \tag{4}$$

where $\|D_{ij}\| = B^{-1}$.

Since the left-hand side of equality (3) is linear with respect to x_i^0, the right-hand side must also remain linear throughout the entire motion. We shall require, in addition, that the right-hand side of equality (3) remain linear under an arbitrary affine deformation of the medium. Multiplying expression (4)

by F_{qi} ($\|F_{ij}\| = \Delta B A^{-1}$) and summing over i, we obtain that the following expression must also be linear with respect to x_k^0:

$$-\frac{1}{\Delta^{\gamma-2}} \frac{1}{\rho^0} \frac{\partial p^0}{\partial x_q^0} + \frac{1}{4\pi} C_{kq} \frac{1}{\rho^0} \frac{\partial H_k^0 H_l^0}{\partial x_l^0} - \frac{1}{8\pi} C_{mn} \frac{1}{\rho^0} \frac{\partial H_m^0 H_n^0}{\partial x_q^0} \quad (\|C_{ij}\| = B^2). \tag{5}$$

Let $C_{ij} = kC_{ij}^0$, where C_{ij}^0 is a constant matrix. Then, differentiating expression (5) with respect to k, we obtain that $\frac{1}{\rho^0} \frac{\partial p^0}{\partial x_i^0}$ are linear functions of x_j^0:

$$\frac{1}{\rho^0} \frac{\partial p^0}{\partial x_i^0} = p_{ij} x_j^0 + p_i \tag{6}$$

(p_{ij} and p_i are constants). Hence it follows that the sum of the last two terms in expression (5) is also a linear function of x_j^0. Differentiating the sum of the last two terms of expression (5) with respect to all C_{ij} that are not identically equal to one another, we again obtain linear expressions in x_j^0, from whose form one may conclude that all

$$\frac{1}{\rho^0} \frac{\partial H_m^0 H_n^0}{\partial x_l^0}$$

are linear functions of x_k^0:

$$\frac{1}{\rho^0} \frac{\partial H_m^0 H_n^0}{\partial x_l^0} = a_{mnlk} x_k^0 + a_{mnl} \tag{7}$$

(a_{mnlk} and a_{mnl} are constants).

In what follows we shall consider the case where at least one of the matrices $\|p_{ij}\|$ or $\|a_{mnlk}\|$ (m and n fixed for the given matrix and varying from one matrix to another) has rank 3. For definiteness, let us assume that this is the matrix $\|p_{ij}\|$. Then from equality (6) one can derive that $p_{ij} = p_{ji}$ and that p^0 is an arbitrary function of the quadratic polynomial Φ:

$$\Phi = p_{ij} x_i^0 x_j^0 + p_{ix} i^0 + \text{const}.$$

For the initial density we obtain the expression

$$\rho^0 = \varkappa p^{0\prime}(\Phi), \tag{8}$$

where $\varkappa$ is a constant, and the prime denotes differentiation with respect to Φ.

If $\rho^0 \neq \text{const}$, then from equalities (7), where ρ^0 must have the value (8), we obtain

$$H_i^0 H_k^0 = \alpha_{ik} p^0 + \beta_{ik}$$

(α_{ik} and β_{ik} are constants). Comparing these equalities with one another, it is not difficult to see that α_{ik} and β_{ik} must be such that the components of the magnetic-field stress will everywhere be proportional to one and the same triple of numbers, i.e. the magnetic lines of force are straight lines.

If $\rho^0 = \text{const}$, then

$$p^0 = \Phi, \qquad H_i^0 H_k^0 = \Phi_{ik}, \tag{9}$$

where Φ and Φ_{ik} are quadratic polynomials in x_j^0, and the Φ_{ik} must satisfy the conditions

$$\Phi_{ik}^2 = \Phi_{ii} \Phi_{kk}$$

(in this equality no summation over repeated indices is implied).

If at least one of the diagonal elements of the matrix Φ_{ik} does not decompose into linear factors or decomposes into factors that are not proportional—

proportional to one another, then from the last equalities it will follow that all diagonal elements must be proportional to it, i.e., the magnetic lines of force again turn out to be straight.

Consequently, the only case in which the magnetic lines of force are not straight is the case in which all Φ_{ii} are squares of linear factors. Then

$$H_i^0 = h_{ik} x_k^0 + h_i \tag{10}$$

(h_{ik} and h_i are constants), and, since $\partial H_k^0/\partial x_k^0 = 0$, we have $h_{kk} = 0$. The magnetic lines of force coincide with the integral curves of the equation

$$\frac{dx_i^0}{dt} = h_{ik} x_k^0 + h_i.$$

Thus, if we do not consider the case in which the magnetic lines of force are straight, then, in order that the motion of the gas proceed according to equalities (1), it is necessary that

$$\rho^0 = \text{const}, \qquad p^0 = p_{ij} x_i^0 x_j^0 + p_{ix} i^0 + \text{const}$$

and that H_i^0 be determined by equalities (10). The $M_{ij}(t)$ and $M_i(t)$ are determined from equation (3), whose coefficients are constant numbers determined from the initial data.

In conclusion, we note that, besides magnetic forces, one could also consider other forces. In particular, stresses depending on the strain tensor and on the strain-rate tensor do not directly affect the motion, since these tensors do not vary from point to point. The influence of such stresses will be manifested only in the form of the heat-inflow equation and in the boundary conditions. Let us consider, for example, viscous stresses. If the viscosity coefficients are constant throughout the medium, then the dissipation of energy per unit volume will also be constant throughout the medium. Since the thermal energy per unit volume of a perfect gas is proportional to the pressure, the additional release of heat due to the dissipation of mechanical energy leads to a uniform increase of the pressure throughout the entire mass of gas and, consequently, has no effect on the motion. If, in addition, $\rho^0 = \text{const}$ and the coefficient of thermal conductivity is constant throughout the medium, then thermal conductivity also has no effect on the motion. Indeed, the pressure, and hence the temperature in this case, are polynomials of second degree in the coordinates, and the release of energy due to thermal conductivity will be uniform in volume.

Since in the case $\rho^0 = \text{const}$ the H_i^0 are expressed linearly in terms of the coordinates, the solutions considered here are at the same time solutions of the equations of magnetohydrodynamics with electrical conductivity different from infinity and constant throughout the medium. Indeed, in this case in the equation

$$\frac{\partial \mathbf{H}}{\partial t} - \operatorname{rot}[\mathbf{v}\mathbf{H}] = \nu_m \Delta \mathbf{H}$$

the last term is equal to zero, and the theorem on the freezing-in of magnetic lines of force is valid despite the finite conductivity. The Joule heat, which in this case is released uniformly in volume, also does not affect the motion of the gas. Thus, if the initial density and the coefficients of thermal conductivity and electrical conductivity are constant in volume, then the solutions indicated in the present work describe motions of a viscous, heat-conducting medium with finite electrical conductivity.

Received
27 XII 1957

REFERENCES

[1] L. I. Sedov, DAN, **90**, No. 5, 735 (1953).
[2] A. G. Kulikovskii, DAN, **114**, No. 5, 984 (1957).
[3] F. Zagar, Atti Acad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat., (8) **18**, 452 (1955).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.