A MIXED PROBLEM FOR AN ELASTIC LAYER

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Abstract

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THEORY OF ELASTICITY

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A MIXED PROBLEM FOR AN ELASTIC LAYER

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In the present paper an exact solution is given of a spatial problem of the theory of elasticity for an unbounded layer ($-\infty < x, y < \infty,\ 0 \le z \le h$), on one boundary plane ($z = 0$) of which elastic displacements (u, v, w) are prescribed, and on the other, stresses $(\sigma_z, \tau_{zx}, \tau_{yz})$.

To solve the problem posed, we represent the displacements u, v, w in terms of the four harmonic functions $(\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ of Papkovich–Neuber ([1,2]).

$$2\mu u = -\frac{\partial F}{\partial x} + 4(1-\nu)\Phi_1, \qquad 2\mu v = -\frac{\partial F}{\partial y} + 4(1-\nu)\Phi_2,$$
$$2\mu w = -\frac{\partial F}{\partial z} + 4(1-\nu)\Phi_3, \qquad F = \Phi_0 + x\Phi_1 + y\Phi_2 + z\Phi_3 \tag{1}$$

(μ is the shear modulus, ν is Poisson’ s ratio).

We shall also give the expressions for the stresses entering the boundary conditions:

$$\sigma_z = 2(1-\nu)\frac{\partial \Phi_3}{\partial z} - \frac{\partial^2 \Phi_0}{\partial z^2} + 2\nu\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right) - \left(x\frac{\partial^2 \Phi_1}{\partial z^2} + y\frac{\partial^2 \Phi_2}{\partial z^2} + z\frac{\partial^2 \Phi_3}{\partial z^2}\right);$$

$$\tau_{zx} = \frac{\partial \Phi}{\partial x} + 2(1-\nu)\frac{\partial \Phi_1}{\partial z}, \qquad \tau_{yz} = \frac{\partial \Phi}{\partial y} + 2(1-\nu)\frac{\partial \Phi_2}{\partial z}, \tag{2}$$

$$\Phi = (1-2\nu)\Phi_3 - \frac{\partial \Phi_0}{\partial z} - \left(x\frac{\partial \Phi_1}{\partial z} + y\frac{\partial \Phi_2}{\partial z} + z\frac{\partial \Phi_3}{\partial z}\right).$$

Using the arbitrariness of one of the harmonic functions entering the solution, we supplement the boundary conditions of the problem

$$u\big|_{z=0} = u_0(r,\varphi), \qquad v\big|_{z=0} = v_0(r,\varphi), \qquad w\big|_{z=0} = w_0(r,\varphi),$$
$$\sigma_z\big|_{z=h} = \sigma(r,\varphi), \qquad \tau_{zx}\big|_{z=h} = \tau_x(r,\varphi), \qquad \tau_{yz}\big|_{z=h} = \tau_y(r,\varphi) \tag{3}$$

(r, φ, z are cylindrical coordinates) by the following two additional conditions:

$$F\big|_{z=0} = 0, \qquad \Phi\big|_{z=h} = 0. \tag{4}$$

In this case, for the functions Φ_1 and Φ_2 one immediately obtains separate boundary conditions:

$$\begin{aligned} 2(1-\nu)\Phi_1\big|_{z=0} &= \mu u_0(r,\varphi), \qquad 2(1-\nu)\frac{\partial \Phi_1}{\partial z}\bigg|_{z=h} = \tau_x(r,\varphi), \\ 2(1-\nu)\Phi_2\big|_{z=0} &= \mu v_0(r,\varphi), \qquad 2(1-\nu)\frac{\partial \Phi_2}{\partial z}\bigg|_{z=h} = \tau_y(r,\varphi). \end{aligned} \tag{5}$$

If the functions Φ_1 and Φ_2 are regarded as found, then we arrive at a boundary-value problem for the harmonic functions Φ_0 and Φ_3 with mixed boundary conditions of the form*

$$\begin{gathered} \Phi_0\big|_{z=0} = -\frac{\mu}{2(1-\nu)}(xu_0 + yv_0) = F_1(r,\varphi), \\ \left[(3-4\nu)\Phi_3 - \frac{\partial \Phi_0}{\partial z}\right]_{z=0} = 2\mu w_0 + \left(x\frac{\partial \Phi_1}{\partial z} + y\frac{\partial \Phi_2}{\partial z}\right)_{z=0} = F_2(r,\varphi), \\ \left[2(1-\nu)\frac{\partial \Phi_3}{\partial z} - z\frac{\partial^2 \Phi_3}{\partial z^2} - \frac{\partial^2 \Phi_0}{\partial z^2}\right]_{z=h} = \sigma + \left(x\frac{\partial^2 \Phi_1}{\partial z^2} + y\frac{\partial^2 \Phi_2}{\partial z^2}\right)_{z=h} \\ -2\nu\left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right)_{z=h} = F_3(r,\varphi), \\ \left[(1-2\nu)\Phi_3 - z\frac{\partial \Phi_3}{\partial z} - \frac{\partial \Phi_0}{\partial z}\right]_{z=h} = \frac{x\tau_x + y\tau_y}{2(1-\nu)} = F_4(r,\varphi). \end{gathered} \tag{6}$$

The exact solution of the mixed problem of potential theory thus obtained can be found by means of the Hankel integral transform.

We shall seek the harmonic functions Φ_0 and Φ_3 in the form of the following expansions in Bessel functions:

$$\begin{aligned} \Phi_0 &= \sum_{n=-\infty}^{\infty} e^{in\varphi} \int_0^\infty (A_0^n \operatorname{ch}\lambda z + B_0^n \operatorname{sh}\lambda z)\, J_n(\lambda r)\, d\lambda, \\ \Phi_3 &= \sum_{n=-\infty}^{\infty} e^{in\varphi} \int_0^\infty (A_3^n \operatorname{ch}\lambda z + B_3^n \operatorname{sh}\lambda z)\, J_n(\lambda r)\, d\lambda. \end{aligned} \tag{7}$$

Substituting (7) into (6) and representing the functions $F_k(r,\varphi)$ ($k = 1,2,3,4$) in the form of the corresponding expansions ([3,4])

$$F_k(r,\varphi) = \sum_{n=-\infty}^{\infty} e^{in\varphi} \int_0^\infty f_k^n(\lambda) J_n(\lambda r)\lambda\, d\lambda,$$
$$f_n^k(\lambda) = \frac{1}{2\pi}\int_0^\infty J_n(\lambda r) r\, dr \int_0^{2\pi} F_k(r,\varphi) e^{-in\varphi}\, d\varphi, \tag{8}$$

we immediately obtain, for the unknown quantities $A_0^n,\ B_0^n,\ A_3^n,\ B_3^n$, a system of linear algebraic equations.

Since from the first equation (6) the quantity $B_0^n = \lambda f_1^n(\lambda)$ is found at once, and from the second equation (6) we have the relation $(3-4\nu)A_3^n - A_0^n = f_2^n(\lambda)$, the matter is in fact reduced to the solution of a system of two equations

$$-\left[(1-2\nu)\operatorname{sh}\gamma + \gamma\operatorname{ch}\gamma\right] A_3^n + \left[2(1-\nu)\operatorname{ch}\gamma - \gamma\operatorname{sh}\gamma\right] B_3^n = \psi_1^n(\gamma),$$

$$-\left[2(1-\nu)\operatorname{ch}\gamma + \gamma\operatorname{sh}\gamma\right] A_3^n + \left[(1-2\nu)\operatorname{sh}\gamma - \gamma\operatorname{ch}\gamma\right] B_3^n = \psi_2^n(\gamma) \tag{9}$$

with determinant

$$D(\gamma) = (3-4\nu)\operatorname{sh}^2\gamma + \gamma^2 + 4(1-\nu)^2 \qquad (\gamma = \lambda h), \tag{10}$$

where the notation has been introduced

$$\psi_1^n = \lambda f_1^n \operatorname{ch}\gamma - f_2^n \operatorname{sh}\gamma + \frac{1}{\lambda} f_3^n, \qquad \psi_2^n = \lambda f_1^n \operatorname{sh}\gamma - f_2^n \operatorname{ch}\gamma + f_4^n. \tag{11}$$

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* It is also required that, as $r \to \infty$, the Papkovich-Neuber functions be of order $1/r$, and their derivatives of order $1/r^2$, which ensures the proper behavior of displacements and stresses at infinity. The functions prescribed in the right-hand sides of (3) must also satisfy the corresponding conditions at infinity.

As an example, let us consider a layer with a fixed base ($z = 0$), deformed by a tangential force T applied at the point $(0, 0, h)$ in the direction of the Ox axis.

Since in the case under consideration $u_0 = v_0 = w_0 = \sigma = \tau_y = 0$, we have $\Phi_2 = 0$, while for Φ_1 we have the boundary conditions

$$\Phi_1\big|_{z=0} = 0, \qquad 2(1-\nu)\frac{\partial \Phi_1}{\partial z}\bigg|_{z=h} = \tau_x(r,\varphi). \tag{12}$$

Distributing the force T over a circle of radius ε, applying the Hankel transform and passing in the obtained solution to the limit as $\varepsilon \to 0$, we find

$$\Phi_1 = \frac{T}{4\pi(1-\nu)} \int_0^\infty \frac{\operatorname{sh} \lambda z}{\operatorname{ch} \lambda h} J_0(\lambda r)\, d\lambda. \tag{13}$$

Substituting the expression found for Φ_1 into the right-hand sides of the basic system (6) and expanding the resulting functions in Hankel integrals, we find the following expressions for the functions Φ_0 and Φ_3:

$$\Phi_0 = \cos\varphi \int_0^\infty A_0^1 \operatorname{ch} \lambda z\, J_1'(\lambda r)\, d\lambda,$$

$$\Phi_3 = \cos\varphi \int_0^\infty \left(A_3^1 \operatorname{ch} \lambda z + B_3^1 \operatorname{sh} \lambda z\right) J_1(\lambda r) \lambda\, d\lambda, \tag{14}$$

where

$$A_0^1 = (3-4\nu) A_3^1 - \frac{Th \operatorname{sh} \gamma}{4\pi(1-\nu) \operatorname{ch}^2 \gamma},$$

and the quantities A_3^1 and B_3^1 must be found from system (9) with the following values of its right-hand sides:

$$\psi_1^1 = -\frac{Th}{4\pi(1-\nu)} \left[1 + (1-2\nu)\frac{\operatorname{th}\gamma}{\gamma}\right], \qquad \psi_2^1 = -\frac{Th}{4\pi(1-\nu)} \operatorname{th}\gamma. \tag{15}$$

Let us give the expression for the tangential stress $\tau_0 \equiv \tau_{zx}|_{z=0}$ in the fixed base:

$$\tau_0 = \frac{Th}{2\pi} \left\{ \frac{1}{r} \int_0^\infty \frac{\operatorname{sh}\gamma}{\operatorname{ch}^2 \gamma} J_1(\lambda r) \lambda\, d\lambda + \int_0^\infty \frac{D_1(\gamma)}{D(\gamma)} \left[\frac{J_1(\lambda r)}{r} \cos 2\varphi - \lambda J_0(\lambda r) \cos^2 \varphi\right] \lambda\, d\lambda \right\}, \tag{16}$$

where the notation has been introduced

$$D_1(\gamma) = 2(1-\nu) \operatorname{sh}\gamma + \frac{\gamma}{\operatorname{ch}\gamma} - \frac{(1-2\nu)^2}{\gamma} \operatorname{th}\gamma \operatorname{sh}\gamma. \tag{17}$$

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Note: Figure translations are in progress. See original paper for figures.