MATHEMATICS

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. L. RASULOV

ON ONE FORMULA FOR THE EXPANSION OF AN ARBITRARY FUNCTION

(Presented by Academician S. L. Sobolev, 20 XI 1957)

In the present note a new formula is given for the expansion of an arbitrary vector-function (see formula (8)), connected with a boundary-value problem with a complex parameter for a system of linear differential equations with piecewise-smooth coefficients. To such a problem there leads the more general problem of investigating the formula for the expansion of an arbitrary vector-function in a series in fundamental* functions of a boundary-value problem with a parameter for a system of linear differential equations of higher order. In addition, the result of this note (see formula (8)) is characterized by the fact that it pertains to a boundary-value problem with a parameter for a system of equations whose coefficients also contain negative powers of λ. As far as we know, such a case has not yet been considered by anyone.

We consider the system of differential equations

$$\frac{dy_k^{(i)}}{dx} - \sum_{j=1}^{n} a_{kj}^{(i)}(x,\lambda) y_j^{(i)} = f_k^{(i)}(x) \quad \text{for } x \in (a_i, b_i) \tag{1}$$

$$(i = 1, \ldots, m;\ k = 1, \ldots, n)$$

with piecewise-smooth coefficients $a_{kj}^{(i)}(x,\lambda)$ under the boundary conditions

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left\{ \alpha_{kj}^{(i)}(\lambda) y_j^{(i)}(a_i) + \beta_{kj}^{(i)}(\lambda) y_j^{(i)}(b_i) \right\} = 0, \tag{2}$$

where

$$a_{kj}^{(i)}(x,\lambda) = \lambda a_{kj}^{(i)}(x) + \sum_{\nu=0}^{N} \lambda^{-\nu} a_{kj\nu}^{(i)}(x);$$

$\alpha_{kj}^{(i)}(\lambda)$, $\beta_{kj}^{(i)}(\lambda)$ are polynomials in λ; (a_i, b_i) are mutually non-overlapping intervals having common ends.

Let the following conditions be satisfied:

1°. On the interval $[a_i, b_i]$ the functions $a_{kj}^{(i)}(x)$ are twice continuously differentiable, $a_{kj0}^{(i)}(x)$ are continuously differentiable once, and the remaining $a_{kj\nu}^{(i)}(x)$ are merely continuous.

2°. For $x \in [a_i, b_i]$ the roots $\varphi_1^{(i)}(x), \dots, \varphi_n^{(i)}(x)$ of the characteristic equations**

$$\Phi^{(i)}(\theta) = \det\left(A^{(i)}(x) - \theta E\right) = 0 \quad (i = 1, \dots, m)$$

* Speaking of fundamental functions, we have in mind the eigenfunctions and associated functions.

** $A^{(i)}(x)$ is the matrix of the functions $a_{kj}^{(i)}(x)$; E is the identity matrix.

are distinct and different from zero; their arguments and the arguments of their differences do not depend on x. For large λ the sign of the differences $\operatorname{Re}\varphi_R^{(i)}(x) - \operatorname{Re}\varphi_j^{(i)}(x)$ does not depend on x.

3°. For $|\lambda| > R$ (R a sufficiently large number) the rank of the matrix of the coefficients $\alpha_{kj}^{(i)}(\lambda), \beta_{kj}^{(i)}(\lambda)$ of the boundary conditions (2) is equal to mn.

Let $y_{k1}^{(i)}(x,\lambda), \dots, y_{kn}^{(i)}(x,\lambda)$ be a fundamental system of particular solutions of the homogeneous system corresponding to system (1). For all λ, with the exception of a countable set of values, the solution $y_k^{(i)}(x,\lambda)$ of problem (1)–(2) exists and is found by the usual method in the form

$$y_k^{(i)}(x,\lambda) = \sum_{p=1}^{n}\sum_{q=1}^{m}\int_{a_q}^{b_q} G_{kp}^{(i,q)}(x,\xi,\lambda) f_p^{(q)}(\xi)\, d\xi,$$

where $G_{kp}^{(i,q)}(x,\xi,\lambda)$ is the Green' s function of problem (1)–(2), which has the form

$$G_{kp}^{(i,q)}(x,\xi,\lambda) = \Delta_{kp}^{(i,q)}(x,\xi,\lambda)/\Delta(\lambda),$$

$\Delta_{kp}^{(i,q)}(x,\xi,\lambda)$, $\Delta(\lambda)$ are analytic functions of λ for $\lambda \neq 0$ (the point at infinity may also be a singular point);

$$\Delta(\lambda) = \begin{vmatrix} u_{1,1}^{(1)} & \cdots & u_{1n}^{(1)} & \cdots & u_{11}^{(m)} & \cdots\, u_{1n}^{(m)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{mn,1}^{(1)} & \cdots & u_{mn,n}^{(1)} & \cdots & u_{mn,1}^{(m)} & \cdots\, u_{mn,n}^{(m)} \end{vmatrix},$$

$$u_{kj}^{(i)} = \sum_{p=1}^{n}\left\{\alpha_{kp}^{(i)}(\lambda) y_{pj}^{(i)}(a_i) + \beta_{kp}^{(i)}(\lambda) y_{pj}^{(i)}(b_i,\lambda)\right\}.$$

According to conditions 1°–3°, the equations

$$\operatorname{Re}\lambda\varphi_p^{(i)}(x) = \operatorname{Re}\lambda\varphi_q^{(i)}(x) \qquad (p, q = 1, \dots, m)$$

for $p \neq q$ determine straight lines passing through the origin of the λ-plane. By these straight lines the λ-plane is divided into sectors (Σ_s), in each of which, under a certain numbering of the roots of the characteristic equations, the inequalities

$$\operatorname{Re}\lambda\varphi_1^{(i)}(x) \leq \operatorname{Re}\lambda\varphi_2^{(i)}(x) \leq \dots \leq \operatorname{Re}\lambda\varphi_n^{(i)}(x). \tag{3}$$

hold. Consequently, according to the theorem of J. D. Tamarkin ([1]), under conditions 1°–3° the homogeneous system corresponding to system (1) has a fundamental system of particular solutions $y_{pq}^{(i)}(x, \lambda)$, admitting in the sectors (Σ_s) the asymptotic representation

$$y_{pq}^{(i)}(x, \lambda) = \exp\left\{\lambda \int_{a_i}^{x} \varphi_q^{(i)}(\xi)\, d\xi\right\} [\eta_{pq}^{(i)}(x)], \tag{4}$$

where $[\eta_{pq}^{(i)}(x)]$ is G. D. Birkhoff' s notation ([2]): the sum

$$\eta_{pq}^{(i)}(x) + E_{pq}^{(i)}(x, \lambda)/\lambda;$$

$\eta_{pq}^{(i)}(x)$, $E_{pq}^{(i)}(x, \lambda)$ are continuous in x on $[a_i, b_i]$; $E_{pq}^{(i)}(x, \lambda)$ is bounded for large $|\lambda|$.

Substituting (4) into the expression $u_{kj}^{(i)}$, we find

$$u_{kj}^{(i)}(\lambda) = \lambda^{l_k}\left\{[A_{kj}^{(i)}] + [B_{kj}^{(i)}]e^{\lambda w_j^{(i)}}\right\}, \tag{5}$$

where $A_{kj}^{(i)}$, $B_{kj}^{(i)}$ are constants;

$$w_j^{(i)} = \int_{a_i}^{b_i} \varphi_j^{(i)}(x)\, dx;$$

l_k is the greatest exponent-

of the powers λ occurring in the k-th row of the matrix of boundary conditions (2) with a nonzero coefficient.

Through the origin and the points $w_1^{(i)}, \dots, w_n^{(i)}$ draw straight lines. Arrange the azimuths α_j of these lines in increasing order. Next define the straight lines d_j by the equations

$$\text{azimuth } d_j = \begin{cases} \dfrac{\pi}{2} - \alpha_j, & \text{if } 0 \leqslant \alpha_j \leqslant \dfrac{\pi}{2}; \\ \dfrac{3\pi}{2} - \alpha_j, & \text{if } \dfrac{\pi}{2} < \alpha_j < \pi. \end{cases}$$

After this we divide the λ-plane into sectors (T_j), not overlapping and having common boundaries (not coinciding with the lines d_j), containing these lines d_j (each such sector contains a half of only one line d_j). Denote by $w_{k_j}^{(i)}$ $(k = 1, \dots, \gamma_j)$ those of the numbers $w_1^{(i)}, \dots, w_n^{(i)}$ which lie on the line with azimuth α_j. For these numbers, on one half of the line d_j contained in (T_j), $\operatorname{Re} \lambda w_{k_j}^{(i)}$ vanishes.

Let $w_{k_j}^{(i)} = \mu_{k_j}^{(i)} \exp\{\alpha_j \sqrt{-1}\}$, where $\mu_{k_j}^{(i)}$ are real numbers arranged in increasing order. After excluding the numbers $w_{k_j}^{(i)}$ from the set $\{w_1^{(i)}, \dots, w_n^{(i)}\}$, the remaining numbers can be divided into two categories, for which in the sector (T_j) we have, respectively: $\operatorname{Re} \lambda w_k^{(i,1)} \to -\infty$, $\operatorname{Re} \lambda w_k^{(i,2)} \to +\infty$ as $\lambda \to \infty$.

Consequently, using the asymptotic formulas (5), we find

$$\Delta(\lambda) = \lambda^l \exp\left\{\lambda \sum w_k^{(i,2)}\right\} H_j(z), \tag{6}$$

where $l = l_1 + \dots + l_{mn}$, the sum extends over all numbers w of the second category for all $i = 1, \dots, m$; $z = \exp\{\alpha_j \sqrt{-1}\}\lambda$,

$$H_j(z) = [M_{1j}] \exp\{m_{1j} z\} + \dots + [M_{\sigma_j j}] \exp\{m_{\sigma_j j} z\};$$

M_{kj}, m_{kj} are constants; $m_{1j} < m_{2j} < \dots < m_{\sigma_j j}$.

With the aid of the asymptotic representation (6), using the method of J. D. Wilder–J. D. Tamarkin ([3,4]), it is easy to show that $\Delta(\lambda)$ has a countable set of zeros, not accumulating at infinity, contained in a finite number of strips of bounded width parallel to the lines d_j and containing them. Further, if these zeros and the point $\lambda = 0$ are isolated by disks of radius δ with centers at the isolated points, then in the remaining part of the λ-plane the inequality holds

$$|H_j(z)| > N_\delta > 0, \tag{7}$$

where N_δ is a constant depending only on δ.

Using the estimate (7), by estimating $\Delta_{pq}^{(i,q)}(x, \xi, \lambda)$ in the sectors (T_j) (with the aid of the asymptotic formulas (4)), by the method of J. D. Tamarkin ([4]) it is not difficult to prove the following theorem.

Theorem. *Under conditions* $1°$–$3°$, *if all the numbers* $M_{1j}, M_{\sigma_j j}$ *are different from zero, then there exists a sequence of closed expanding contours* Γ_ν $(\nu = 1, 2, \ldots)$ *such that for every vector-function* $f^{(i)}(x)$

* These contours are chosen so that they do not intersect the above-mentioned disks of radius δ with centers at the isolated points.

with components $f_k^{(i)}(x) \in L_2(a_i, b_i)$ $(i = 1, \ldots, m;\ k = 1, \ldots, n)$, the integral

$$-\frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_\lambda^\nu} y^{(i)}(x, \lambda)\, d\lambda \Rightarrow (A^{(i)}(x))^{-1} f^{(i)}(x) \quad \text{as } \nu \to \infty \tag{8}$$

in the sense of $L_2(a_i, b_i)$.*

We note that the formulation of this theorem for $m = 1$ is contained in (5).

Lviv State University
named after Ivan Franko

Received
18 XI 1957

REFERENCES

[1] Ya. D. Tamarkin, On some general problems in the theory of ordinary linear differential equations and on expansions of arbitrary functions in series, Petrograd, 1917.
[2] G. D. Birkhoff, Trans. Am. Math. Soc., 9 (1908).
[3] Ch. E. Wilder, Trans. Am. Math. Soc., 18 (1917); 19 (1918).
[4] J. Tamarkin, Math. Zs., 27 (1928).
[5] M. L. Rasulov, Matem. sborn., 30 (72), 3 (1952).

* By imposing stronger restrictions on $f^{(i)}(x)$, one can obtain convergence in the pointwise sense.

Note: Figure translations are in progress. See original paper for figures.