AN INVERSION FORMULA FOR THE SOMMERFELD INTEGRAL

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Abstract

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As is known, a solution of the two-dimensional wave equation

$$\Delta S - m^2 S = 0, \qquad \left(-\frac{\pi}{2} \leqslant \arg m \leqslant \frac{\pi}{2}\right) \tag{1}$$

of the form

$$S(r,\varphi) = \frac{1}{2\pi i}\int_\gamma e^{mr\cos\alpha} s(\alpha+\varphi)\,d\alpha \tag{2}$$

was used by Sommerfeld [1] for a rigorous treatment of the diffraction of a plane wave by a wedge under the boundary conditions $S = 0$ or $\partial S/\partial n = 0$. The systematic method, recently proposed [2], for finding the function $s(\alpha)$ made it possible to solve the same diffraction problem for the boundary conditions

$$\partial S/\partial n + hS = 0.$$

Theorem 1, proved in the present article, makes it possible to find, in the form of the Sommerfeld integral (2), the solution of certain new boundary-value problems in wedge-shaped regions.

Theorem 1. *Let M, a, b, c, d be positive numbers; let ε, m be numbers satisfying the conditions: $0 < \varepsilon < \pi$, $|\arg m| \leqslant \pi/2$. An integral equation is given*

$$F(r) = \frac{1}{2\pi i}\int_\gamma e^{mr\cos\alpha} f(\alpha)\,d\alpha. \tag{3}$$

The prescribed function $F(r)$ satisfies the inequality $|F(r)| < M|r|^{-1+a}e^{b|r|}$ for positive values of r, and also in the whole domain $c < |r| < \infty$, $|\arg r| < \varepsilon_1$, where this function is analytic and regular.

The contour of integration γ (Fig. 1) consists of two loops. The loop γ_1 consists of two half-lines $\operatorname{Re}\alpha = \arg m \pm (\varepsilon + \pi/2)$, $\operatorname{Im}\alpha \geqslant d$, and of the segment of the

Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

straight line $\operatorname{Im}\alpha = d$. *The loop* γ_2 *is symmetric to* γ_1 *with respect to the origin* $\alpha = 0$.

Then, among analytic functions $f(\alpha)$*, regular on the contour* γ *and inside both loops everywhere except, possibly, infinitely distant points, and satisfying in these regions the inequality* $|f(\alpha)| < M_1 \exp[(1-a_1)|\operatorname{Im}\alpha|]$*, there exists a unique odd function that is a solution of the integral equation (3). For* $\operatorname{Re}(m\cos\alpha) > b$ *this function is represented by the integral*

$$f(\alpha) = -\frac{m\sin\alpha}{2}\int_0^\infty F(r)e^{-mr\cos\alpha}\,dr. \tag{4}$$

For this function $a_1 = a$.

Proof. For an odd function $f(\alpha)$, using the symmetry of the contour γ, one can reduce the contour to a single loop γ_1, rewriting (3) in the form

$$F(r) = \frac{1}{\pi i}\int_{\gamma_1} e^{mr\cos\alpha} f(\alpha)\,d\alpha. \tag{5}$$

If we introduce the function

$$g(w) = -2f(\alpha)\exp(-i\arg m)/\sin\alpha,$$

where $w = \exp(i\arg m)\cos\alpha$, then integral (5) takes the form

$$F(r) = \frac{1}{2\pi i}\int_\Gamma e^{|m|rw} g(w)\,dw, \tag{6}$$

where the contour Γ, into which γ_1 passes (Fig. 2), intersects the real axis between zero and $\operatorname{ch} d$ and at infinity coincides with the rays $\arg w = \pm(\varepsilon + \pi/2)$.

Fig. 1 Fig. 2

The function $g(w)$ is regular in the domain to the right of the contour Γ, since this domain is the image of the interior of the loop γ_1, where the function $f(\alpha)/\sin\alpha$ is regular. The order of decrease $|g(w)| < 4M_1|w|^{-a_1}$ as $|w| \to \infty$, $|\arg w| < \varepsilon + \pi/2$, is obtained from the restriction imposed on the order of growth of the admissible functions $f(\alpha)$.

To find the function $g(w)$ from equation (6), multiplying by $\exp(-|m|rw)$, where $\operatorname{Re} w > \operatorname{ch} d$, and integrating with respect to r, we have

$$\int_0^\infty F(r)e^{-|m|rw}\,dr = \frac{1}{2\pi i}\int_{+\theta}^{+\infty} dr \int_\Gamma e^{|m|r(w_1-w)} g(w_1)\,dw_1.$$

Changing the order of integration on the right-hand side and then passing to the limit, we find

$$\int_0^\infty F(r)e^{-|m|rw}\,dr = -\frac{1}{2\pi i|m|}\int_\Gamma \frac{g(w_1)}{w_1 - w}\,dw_1 = \frac{g(w)}{|m|}, \tag{7}$$

where the integral over the contour Γ has been reduced to the residue by virtue of the condition $a_1 > 0$. Hence, returning again to the function $f(\alpha)$, we obtain the required odd solution (4) of the integral equation (3).

To verify that the regularity assumed for $f(\alpha)$ does indeed hold for the solution, we note that in integral (7) the portion of the path of integration $r > c$, owing to the analyticity of $F(r)$, can be brought in the complex plane arbitrarily close to any of the half-lines $|\arg r| = \varepsilon_1 > \varepsilon$, after which one can verify the regularity of the function $g(w)$ for large $|w|$ throughout the interval $|\arg w| < \varepsilon + \pi/2$, and consequently the regularity of $f(\alpha)/\sin\alpha$ for large $\operatorname{Im}\alpha > d$ in the strip $|\operatorname{Re}\alpha - \arg m| < \varepsilon + \pi/2$, corresponding to the interior of the loop γ_1.

Finally, using the condition $|F(r)| < M|r|^{-1+a}e^{b|r|}$, from expression (7) we obtain

$$|g(w)| < \int_0^\infty |F(r)|e^{-r|wm|\cos(\arg w)}\,dr < M|mw|^{-a}\int_0^\infty e^{-x[\cos(\arg w) - b/|mw|]}x^{1-a}\,dx.$$

Since the integral on the right for large values of $|w|$ represents a bounded quantity, we obtain the estimate $|g(w)| < M_2|w|^{-a}$. Consequently, for the solution $f(\alpha)$ one may take $a_1 = a$.

Thus, Theorem 1 is proved. Expressions (3) and (4) are inversion formulas for the Sommerfeld integral.

Uniqueness, following from the fact that, by virtue of (4), an odd solution $f(\alpha)$ of the homogeneous integral equation

$$\frac{1}{2\pi i}\int_\gamma e^{mr\cos\alpha} f(\alpha)\,d\alpha = 0 \tag{8}$$

identically equal to zero, is essentially connected with the condition $f(\alpha) = O\{\exp[(1-a)|\operatorname{Im}\alpha|]\}$ $(a > 0)$. This is seen from the following theorem.

Theorem 2. Let $f(\alpha)$ be an analytic function, regular on the contour γ and inside the loops γ_1 and γ_2 indicated in Theorem 1, except for infinitely remote points. As $|\operatorname{Im}\alpha| \to \infty$ in these regions,

$$f(\alpha) = O\{\exp[(n+1-a)|\operatorname{Im}\alpha|]\},$$

where $0 < a < 1$ and n is a positive integer or zero.

Then, in order that the identity (8) hold for $r > 0$, it is necessary and sufficient that the function $f(\alpha)$ have the form

$$f(\alpha) = f_1(\alpha) + \sin\alpha \sum_{\nu=0}^{n} c_\nu \cos^{\nu-1}\alpha, \tag{9}$$

where $f_1(\alpha)$ is an arbitrary even function and the coefficients c_ν are arbitrary constants, or, as a consequence of (9), that the function $f(\alpha)$ satisfy the functional equation

$$f(\alpha) - f(-\alpha) = 2\sin\alpha \sum_{\nu=0}^{n} c_\nu \cos^{\nu-1}\alpha. \tag{10}$$

Proof. Substitution of the even function $f_1(\alpha)$ turns the integral (8) into zero owing to the symmetry of the contour γ. We denote the remaining odd part of the function $f(\alpha)$, equal to $\frac{1}{2}[f(\alpha) - f(-\alpha)]$, by $f_2(\alpha)$. By means of n-fold integration by parts we can reduce the integral equation (8) to the form

$$\frac{1}{2\pi i}\int_\gamma e^{mr\cos\alpha} D^n[f_2(\alpha)]\,d\alpha = 0,$$

where D is an operation of the form

$$D[f(\alpha)] = \frac{1}{m}\frac{d}{d\alpha}\left[\frac{f(\alpha)}{\sin\alpha}\right].$$

In view of the fact that the odd function $D^n[f_2(\alpha)]$, as $|\operatorname{Im}\alpha| \to \infty$, has order $\exp[(1-a)|\operatorname{Im}\alpha|]$, the conditions of Theorem 1 are now satisfied, and therefore, by virtue of (4), we have $D^n[f_2(\alpha)] = 0$. Hence, by n-fold integration we obtain an expression containing n arbitrary constants, which can be written in the form

$$f_2(\alpha) = \sin\alpha \sum_{\nu=0}^{n} c_\nu \cos^{\nu-1}\alpha.$$

Thus the necessity and sufficiency of each of the conditions (9), (10) is evident, and Theorem 2 is proved.

Let us note that the formula

$$D^n[f(\alpha)] = -\frac{m\sin\alpha}{2}\int_0^\infty e^{-mr\cos\alpha} F(r) r^n\,dr \tag{11}$$

can serve as the inverse to (3) in the case when, as $r \to 0$, the function $F(r)$ has order of growth $|r|^{a-(n+1)}$.

Theorem 2 makes it possible to obtain solutions of boundary-value problems in wedge-shaped domains when arbitrary derivatives of any order occur in the boundary conditions; in particular, the problem of diffraction of sound waves by a semi-infinite elastic plate. The arbitrary constants acquire in this problem a simple meaning.

In all the formulas given above, instead of the wave number k, following V. A. Fock, we introduced the complex quantity m. When considering wave problems for media without absorption and with positive phase velocity[3], taking the time dependence according to the factor $\exp(-i\omega t)$, one may put $m = -ik$, regarding k as a positive number. Then the Sommerfeld integral (2) and the inversion formulas (3), (4) take the form

$$S(r,\varphi) = \frac{1}{2\pi i}\int_\gamma e^{-ikr\cos\alpha} s(\alpha+\varphi)\,d\alpha; \tag{12}$$

$$F(r) = \frac{1}{2\pi i}\int_\gamma e^{-ikr\cos\alpha} f(\alpha)\,d\alpha; \tag{13}$$

$$f(\alpha) = \frac{ik\sin\alpha}{2}\int_0^\infty e^{ikr\cos\alpha}F(r)\,dr \tag{14}$$

and the paths of integration for the two loops of the contour γ pass between the limits: $\gamma_1(i\infty+\varepsilon, i\infty-\pi-\varepsilon)$ and $\gamma_2(-i\infty-\varepsilon, -i\infty+\pi+\varepsilon)$.

As the simplest example of the application of the inversion formula (14) in combination with the method developed earlier ([2,4]), let us find the forced oscillation ([5]) $S(r,\varphi)$, satisfying the equation $\Delta S + k^2 S = 0$ in the wedge-shaped region $-\Phi < \varphi < \Phi$, excited by the combined action of the wave $S_0\exp[-ikr\cos(\varphi-\varphi_0)]$ ($|\operatorname{Re}\varphi_0| < \Phi-\varepsilon$), incident from infinity, and of sources distributed on the faces of the wedge, determined by prescribing the functions $F_\pm(r)$ in the boundary conditions $-\partial S/\partial\varphi + ikrF_\pm(r) = 0$ ($\varphi = \pm\Phi$), with $|F_\pm(r)| < \infty$ for $r < \infty$.

Then we shall find the required unique, continuous and single-valued (including the boundary points) solution in the form of the Sommerfeld integral (12), if we seek the function $s(\alpha)$ among functions satisfying the following conditions: 1) the function $s(\alpha)-S_0/(\alpha-\varphi_0)$ is regular in the strip $|\operatorname{Re}\alpha| < \Phi$; 2) in this strip $|s(\alpha) - s(\pm i\infty)| < M\exp(-a|\operatorname{Im}\alpha|)$ as $\operatorname{Im}\alpha \to \pm\infty$; 3) $s(-i\infty) = -s(i\infty) = iS(0,\varphi)/2$ ($S(0,\varphi)$ does not depend on φ).

Represent the two terms in the left-hand side of the boundary conditions in the form of Sommerfeld integrals

$$-\frac{1}{ikr}\left.\frac{\partial S}{\partial \varphi}\right|_{\varphi=\pm\Phi} = \frac{1}{2\pi i}\int_\gamma e^{-ikr\cos\alpha}\frac{\sin\alpha}{2}\left[s(\pm\Phi+\alpha)+s(\pm\Phi-\alpha)\right]d\alpha,$$

$$F_\pm(r) = \frac{1}{2\pi i}\int_\gamma e^{-ikr\cos\alpha} f_\pm(\alpha)\,d\alpha,$$

where the functions $f_\pm(\alpha)$ are expressed in terms of the prescribed functions $F_\pm(r)$ by the inversion formula (14). Then, on the basis of theorem 2 and conditions 2) and 3), we obtain two functional equations $s(\alpha \pm \Phi) + s(-\alpha \pm \Phi) = -2f_\pm(\alpha)/\sin\alpha$. The solution of these equations, having a single pole with principal part $S_0/(\alpha-\varphi_0)$ in the strip $|\operatorname{Re}\alpha| < \Phi+\varepsilon$, is readily found with the aid of the Fourier integral

$$S(\alpha) = S_0\frac{\pi}{2\Phi}\cos\frac{\pi\alpha}{2\Phi}\Big/\left[\sin\frac{\pi\alpha}{2\Phi} - \sin\frac{\pi\varphi_0}{2\Phi}\right] +$$

$$+\frac{i}{2\sqrt{2\pi}}\left(\int_{-i\infty-\delta}^{i\infty-\delta} + \int_{-i\infty+\delta}^{i\infty+\delta}\right)\frac{R_+(w)e^{-iw\Phi} - R_-(w)e^{iw\Phi}}{i\sin 2w\Phi}\,e^{-iw\alpha}\,dw, \qquad (15)$$

where

$$R_\pm(w) = \frac{i}{\sqrt{2\pi}}\int_{-i\infty}^{i\infty}\frac{f_\pm(\alpha)}{\sin\alpha}\,e^{iw\alpha}\,d\alpha.$$

Formulas (12), (15) give the solution of the problem. In the particular case $F_\pm(r) \equiv 0$, when in (15) only the first term remains, one obtains the well-known Sommerfeld solution of the problem of diffraction of a plane wave by a perfectly rigid wedge. The opposite particular case $S_0 = 0$ is the solution of the two-dimensional Neumann boundary-value problem in a wedge-shaped region for the equation $\Delta S + k^2 S = 0$.

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Note: Figure translations are in progress. See original paper for figures.