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Abstract

Full Text

MATHEMATICAL PHYSICS

M. V. MASLENNIKOV

ON WICK' S PROBLEM

(Presented by Academician M. V. Keldysh, 15 I 1958)

We consider the stationary problem of the slowing down and diffusion of neutrons emitted by a plane source in an infinite homogeneous isotropic medium with moderator-nucleus mass M (the neutron mass is taken as unity), whose parameters do not depend on the neutron energy. It is assumed that the thermal motion of the nuclei and their chemical bonds are negligibly small and that neutron scattering by nuclei is elastic and isotropic in the center-of-mass system of the neutron and the nucleus*. This problem has been studied by a number of authors (¹⁻⁶), who obtained a well-known asymptotic formula for the space-energy distribution of the neutron-collision density. From the mathematical point of view, its solution is carried furthest in Wick' s work (⁴). However, the aim of these works is mainly to develop methods for computing the neutron-collision density on the basis of the aforementioned asymptotic formula. At the same time, the derivation itself of this formula suffers from mathematical gaps**. Below we present the main points of a rigorously carried out solution of Wick' s problem (ending with the derivation of the asymptotic formula). As in (¹⁻⁶), the starting point is the Fourier-Laplace transform of the basic equation (1). But in studying the transformed equation it is possible to avoid using the method of continued fractions employed in (¹⁻⁶). In addition, the problem is generalized in the following directions: nonisotropic sources are included in the consideration; the asymptotics of the collision density

$$\psi(z, u, \mu)$$

as a whole is sought, and not only that of its moment

$$\psi_0(z, u) = 2\pi \int_{-1}^1 \psi(z, u, \mu) d\mu.$$

Let sources of nonzero strength have a plane-parallel structure and be concentrated in a layer orthogonal to the Cartesian axis Oz ; μ is the cosine of the angle between the direction $\vec{\Omega}$ of the neutron velocity and Oz ; u is the neutron lethargy. We shall assume that the phase density of the sources $S = S(z, u, \mu)$ satisfies the following restrictions: 1) $S(z, u, \mu) = 0$, if $|z| \geq z_0 > 0$ or $u \geq u_0 > 0$; 2) $0 \leq S(z, u, \mu) \leq S^+ = \text{const} < \infty$,

$$z \in (-\infty, \infty), \quad u \geq 0, \quad \mu \in [-1, 1];$$

3) the total variation

$$\bigvee_{u=0}^{u_0} S(z, u, \mu) \leq S_1^+ = \text{const} < \infty, \quad z \in (-\infty, \infty), \quad \mu \in [-1, 1].$$

We take the total mean free path to be unity; the probability of scattering in one collision is denoted by h , $h > 0$. $\psi(z, u, \mu)$ satisfies the equation

$$\psi(z, u, \mu) = h\hat{A}\psi(z, u, \mu) + \int_0^\infty e^{-\rho} S(z - \rho\mu, u, \mu) d\rho; \quad (1)$$

* Nothing essential changes if one includes in the consideration anisotropy of scattering not depending on the neutron energy.

** Wick ⁽⁴⁾ emphasizes the necessity of a mathematically rigorous treatment.

$$\begin{aligned} \hat{A}F(z, u, \mu) = & \frac{\alpha}{\pi} \int_0^\infty e^{-\rho} d\rho \int_{\vec{\Omega}\vec{\Omega}' > f(u)} F(z - \rho\mu, u - \vartheta(\vec{\Omega}\vec{\Omega}'), \vec{\Omega}') \times \\ & \times e^{-\nu(\vec{\Omega}\vec{\Omega}')} \frac{d\vec{\Omega}'}{[(\vec{\Omega}\vec{\Omega}')^2 + M^2 - 1]^{1/2}}; \end{aligned} \quad (2)$$

$$\vartheta(x) = 2 \ln \frac{M+1}{x + (x^2 + M^2 - 1)^{1/2}}; \quad q = \vartheta(-1); \quad \alpha = (1 - e^{-q})^{-1};$$

$$f(u) = \frac{M+1}{2} e^{-u/2} - \frac{M-1}{2} e^{u/2} \text{ for } u \in [0, q); \quad f(u) = -1 \text{ for } u \geq q.$$

As shown in (7), in a sufficiently broad class of functions equation (1) has a unique solution $\psi(z, u, \mu)$, and everywhere

$$\psi(z, u, \mu) \leq L \exp(-\varkappa|z| + (p_0 - 1)u). \quad (3)$$

Here $\varkappa \in [0, 1)$; $p_0 > p_0(\varkappa)$; $p_0(\varkappa)$ is the real root of the equation $\alpha h(1 - \varkappa)^{-1}(p_0(\varkappa))^{-1}(1 - \exp(-qp_0(\varkappa))) = 1$; L is a sufficiently large constant. Put

$$\psi_1(z, u, \mu) = \psi(z, u, \mu) - (h\hat{A} + 1) \int_0^\infty e^{-\rho} S(z - \rho\mu, u, \mu) d\rho; \quad (4)$$

$$\Phi(k, \eta, \mu) = \int_{-\infty}^{\infty} e^{kz} dz \int_0^{\infty} e^{-\eta u} \psi_1(z, u, \mu) du; \quad (5)$$

$$\sigma(k, \eta, \mu) = \int_{-\infty}^{\infty} e^{kz} dz \int_0^{\infty} e^{-\eta u} S(z, u, \mu) du.$$

Then from (1) one obtains the equation

$$(1 - k\mu)\Phi(k, \eta, \mu) = h\hat{G}(\eta)\Phi(k, \eta, \mu) + h\hat{G}(\eta)(1 - k\mu)^{-1}h\hat{G}(\eta)\sigma(k, \eta, \mu)(1 - k\mu)^{-1}, \quad (6)$$

$$\hat{G}(\eta)F(\mu) = \int_{-1}^1 g_{\eta}(\mu, \mu')F(\mu') d\mu';$$

$$g_{\eta}(\mu, \mu') = \frac{\alpha}{\pi} \int_0^{2\pi} \left(\frac{\mu_0 + (\mu_0^2 + M^2 - 1)^{1/2}}{M + 1} \right)^{2\eta+2} \frac{d\varphi}{(\mu_0^2 + M^2 - 1)^{1/2}}, \quad \text{if } M > 1,$$

$$g_{\eta}(\mu, \mu') = \frac{1}{\pi} \int_{\mu_0 > 0} \mu_0^{2\eta+1} d\varphi, \quad \text{if } M = 1;$$

$$\mu_0 = \mu\mu' + \sqrt{1 - \mu^2}\sqrt{1 - \mu'^2} \cos \varphi;$$

$\sigma(k, \eta, \mu)$ is an entire function of (k, η) ; $\Phi(k, \eta, \mu)$ is analytic in (k, η) and satisfies equation (6) for

$$(k, \eta) \in Q = \{(k', \eta') \mid |\operatorname{Re} k'| < 1, \operatorname{Re} \eta' > p_0(|\operatorname{Re} k'| - 1)\}.$$

Equation (6) makes it possible to carry out the analytic continuation of $\Phi(k, \eta, \mu)$ to a broader domain.

Let

$$E = \{k \mid |\operatorname{Im} k| > 0 \text{ or } \operatorname{Im} k = 0, \text{ but } |\operatorname{Re} k| < 1\},$$

and let H be the whole complex η -plane if $M > 1$; $H = \{\eta \mid \operatorname{Re} \eta > -1\}$ if $M = 1$. Introduce the operator $\hat{G}(k, \eta)$ by the rule

$$\hat{G}(k, \eta)F(\mu) = (1 - k\mu)^{-1/2}\hat{G}(\eta)(1 - k\mu)^{-1/2}F(\mu),$$

and let \hat{E} be the identity operator, while L_p is the space of complex-valued functions of the variable μ , summable with the p -th power

module on $[-1, 1]$. Fix $(k_1, \eta_1) \in E \times H$ so that the operator $\hat{E} - h\hat{G}(k_1, \eta_1)$ is invertible in the space $(L_p \rightarrow L_p)$ of linear bounded operators mapping L_p into L_p , for some $p > 1$ (if $M = 1$, then let also $p > \frac{1}{2}(1 + \operatorname{Re} \eta_1)^{-1}$). Then it can be shown that (6) is uniquely solvable in some neighborhood of (k_1, η_1) , and in this neighborhood $\Phi(k, \eta, \mu)$ is bounded for all values of its arguments, analytic in (k, η) , and continuous in $\mu \in [-1, 1]$. Thus $\Phi(k, \eta, \mu)$ is analytically continued to the domain Q_1 , obtained by adjoining to Q all (k, η) for which $\hat{E} - h\hat{G}(k, \eta)$ is invertible.

To clarify the nature of the singularities of $\Phi(k, \eta, \mu)$, in the space L_2 the homogeneous equation

$$M(k, \eta)\Phi_{k, \eta}(\mu) = h\hat{G}(k, \eta)\Phi_{k, \eta}(\mu) \quad (7)$$

is studied, first for real k and η , $(k, \eta) \in E \times H$. Using the theory of cones in Banach spaces and the perturbation theory of bounded operators, one can establish the existence of exactly one nonnegative function $\Phi_{k, \eta}(\mu)$ satisfying equation (7). Here $M(k, \eta)$ turns out to be a simple positive eigenvalue, and $\Phi_{k, \eta}(\mu)$ an everywhere positive continuous function of μ . $M(k', \eta')$ and $\Phi_{k', \eta'}(\mu)$ are expanded in some neighborhood of the point (k, η) into absolutely and uniformly convergent series in powers of $(k' - k)$ and $(\eta' - \eta)$, with coefficients continuously depending on μ . At all points of this neighborhood equation (7) is satisfied. Studying $M(k, \eta)$, one can show that for $\eta \geq \eta_0 = p_0(0) - 1$ there exists exactly one solution $k = k(\eta) \in [0, 1)$ of the equation $M(k, \eta) = 1$, and $k(\eta)$ turns out to be a regular function of η for $\eta > \eta_0$. For $\eta > \eta_0$, $k'(\eta) > 0$, $k''(\eta) < 0$.

Now one can determine the structure of Q_1 . Rather cumbersome arguments lead, in particular, to the following conclusions. Let $\gamma_{1,2}$ and $p > 2$ be fixed numbers, $\gamma_2 > \gamma_1 > \eta_0$ ($p > \frac{1}{2}(1 + \eta_0)^{-1}$, if $M = 1$), and let $\Sigma = \{\eta \mid \operatorname{Re} \eta \in [\gamma_1, \gamma_2]\}$. There exists $\sigma_1 > 0$ such that $k(\eta)$ is analytic on $\Sigma_1 = \{\eta \in \Sigma, \mid \operatorname{Im} \eta \mid \leq \sigma_1\}$. To each point $\eta \in \Sigma$ there is put in correspondence a contour $\mathcal{L}_j(\operatorname{Re} \eta)$ ($j = 1$ for $\eta \in \Sigma_1$, $j = 2$ for $\eta \in \Sigma \setminus \Sigma_1$), symmetric with respect to the real axis and consisting of three rectilinear links with vertices at the points $k(\operatorname{Re} \eta) + \varepsilon_j \pm is$. Along $\mathcal{L}_j(\operatorname{Re} \eta)$, $\operatorname{Im} k$ changes monotonically. The outer links of $\mathcal{L}_j(\operatorname{Re} \eta)$ are rays issuing from the vertices and suitably inclined in the direction of increasing $\operatorname{Re} k$ ($\varepsilon_{1,2}$ and s are fixed positive numbers). For fixed $\eta \in \Sigma$ and $\mu \in [-1, 1]$, $\Phi(k, \eta, \mu)$ is analytic in k in the closed domain with boundaries $\operatorname{Re} k = 0$ and $\mathcal{L}_j(\operatorname{Re} \eta)$ everywhere, except for the point $k = k(\eta)$, which lies inside this domain for $j = 1$. From 1)–3) and (6) it can be derived that for $\eta \in \Sigma$, $\mu \in [-1, 1]$, and $z \geq z_1 > z_0$,

$$\left| \frac{1}{2\pi i} \int_{\mathcal{L}_j(\operatorname{Re} \eta)} e^{-kz} \Phi(k, \eta, \mu) dk \right| \leq K e^{-z(k(\operatorname{Re} \eta) + \varepsilon_j)} \frac{\mathcal{N}_{p, p_1}(\eta)}{1 + \mid \operatorname{Im} \eta \mid}. \quad (8)$$

Here $K > 0$ and does not depend on z, η , and μ ; $p_1 \in (2, p]$, and, if $M = 1$, $p_1(\frac{1}{2} + \eta_0) > -1$;

$$\varkappa_{p,p_1}(\eta) = \left\{ \int_{-1}^1 \int_{-1}^1 |g_\eta(\mu, \mu')|^{p_1} d\mu d\mu' \right\}^{1/p}. \quad (9)$$

The function $\varkappa_{p,p_1}(\eta)$ is finite and continuous for $\operatorname{Re} \eta \geq \eta_0$. There exists a constant $D > 0$ such that, for $\eta_1 > \eta_0$,

$$\left\{ \int_{-\infty}^{\infty} \varkappa_{p,p_1}^p(\eta_1 + it) dt \right\}^{1/p} \leq D. \quad (10)$$

The proof of this estimate is based on Titchmarsh' s theorem on Fourier transforms in the classes L_r with $r \in (1, 2]$.

Upon inversion of the Fourier transform entering into (5), the integral

$$\frac{1}{2\pi i} \int_{\operatorname{Re} k=0} e^{-kz} \Phi(k, \eta, \mu) dk \quad (11)$$

is considered for $z \geq z_1$, $\eta \in \Sigma$. It is not difficult to establish the applicability of Jordan's lemma, so that the integration in (11) can be reduced to integration over $\mathcal{L}_j(\operatorname{Re} \eta)$, taking account of the corresponding residue at $k = k(\eta)$ (if $\eta \in \Sigma_1$). This residue is expressed through $\varphi_\eta(\mu) \equiv (1 - k(\eta)\mu)^{-1/2} \Phi_{k(\eta), \eta}(\mu)$ (cf. (7)). To find it, $\Phi(k, \eta, \mu)$ is expanded in a Laurent series in powers of $k - k(\eta)$, which can be done with the aid of (6). The resulting expression for the integral (11) is substituted into the inversion of the Laplace integral from (5):

$$\begin{aligned} \psi_1(z, u, \mu) &= \frac{1}{2\pi i} \int_{\gamma - i\sigma_1}^{\gamma + i\sigma_1} \frac{(\sqrt{1 - k(\eta)\mu} \sigma_1(k(\eta), \eta, \mu), \overline{\varphi_\eta(\mu)})}{(\varphi_\eta^2(\mu), \mu)} \times \\ &\times \varphi_\eta(\mu) e^{-zk(\eta) + \eta u} d\eta + \psi^{(\gamma)}(z, u, \mu). \end{aligned} \quad (12)$$

Here $\gamma \in [\gamma_1, \gamma_2]$; $\sigma_1(k, \eta, \mu) = h^2 \hat{G}^2(k, \eta)(1 - k\mu)^{-1/2} \sigma(k, \eta, \mu)$; (a, b) is the scalar product of vectors $a, b \in L_2$;

$$|\psi^{(\gamma)}(z, u, \mu)| \leq \frac{K}{2\pi} e^{\gamma u - z(k(\gamma) + \varepsilon_3)} \int_{-\infty}^{\infty} \frac{x_{p,p_1}(\gamma + it)}{1 + |t|} dt \leq K_1 e^{\gamma u - z(k(\gamma) + \varepsilon_3)}, \quad (13)$$

$$K_1 = \text{const} < \infty \quad (\text{cf. (10)}); \quad \varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\} > 0.$$

Suppose that z and u are positive, and that the saddle point γ_0 , uniquely determined by the relation $k'(\gamma_0) = u/z$, is contained in $[\gamma_1, \gamma_2]$. Put $\gamma = \gamma_0$ in (12) and extract from the integral the principal part by the method of steepest descents, calculating the first two terms of the asymptotic series and estimating the remainder in the known way⁸, uniformly with respect to $u/z \in [x_1, x_2]$, $z \geq z_1$, $\mu \in [-1, 1]$ ($x_1 = k'(\gamma_2)$, $x_2 = k'(\gamma_1)$). Thus the principal term of $\psi_1(z, u, \mu)$ is obtained, which (cf. (4)) coincides with the asymptotics of $\psi(z, u, \mu)$. Analyzing the set of phase points (z, u, μ) for which (12) and (13) are valid, one can single out a phase set \mathfrak{M} of zero measure such that the following final result proves valid:

Let arbitrary positive numbers x_1, x_2 and $z_1, x_2 > x_1$, $z_1 > z_0$, be given. There exists a finite positive constant A such that, for all points (z, u, μ) satisfying the conditions $z \geq z_1$, $u/z \in [x_1, x_2]$, $\mu \in [-1, 1]$, $(z, u, \mu) \notin \mathfrak{M}$, the relations

$$\psi(z, u, \mu) = \frac{(\sigma(k(\gamma_0), \gamma_0, \mu), \varphi_{\gamma_0}(\mu))}{(2\pi z |k''(\gamma_0)|)^{1/2} (\mu, \varphi_{\gamma_0}^2(\mu))} \varphi_{\gamma_0}(\mu) e^{-zk(\gamma_0) + u\gamma_0} \left(1 + \frac{f(z, u, \mu)}{z} \right);$$

$$|f(z, u, \mu)| \leq A; \quad k'(\gamma_0) = \frac{u}{z}.$$

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Division of Applied Mathematics
V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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