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Abstract

Full Text

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ON THE THEORY OF OPERATIONAL CALCULUS

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In 1950-1951 Mikusiński ([1]) constructed an operational calculus without any connection with the theory of the Laplace transform. Despite all the attractiveness and elegance of this theory, it must be said that in a number of cases, when the Laplace integral is used, various transformations and computations connected with the finding of operational formulas are considerably simplified. Therefore it is expedient to construct, for Mikusiński' s operator calculus, an analogue of the Laplace transform. Such a construction is given in this note. It should be regarded as a generalization of our paper ([2]).

Denote by L the set of all functions defined on the interval $0 < t < \infty$ and summable in the Lebesgue sense on every finite interval $(0, A)$, $A > 0$. Let S be the set of all functions in L for which the Laplace integral

$$\bar{f}(p) = \int_0^\infty f(t)e^{-pt}, \qquad p = \sigma + i\tau, \tag{1}$$

converges absolutely, and let $\bar{S}$ be the set of their Laplace transforms, i.e., the totality of all functions $\bar{f}(p)$, $f(t) \in S$.

From the known properties of the Laplace integral it follows that $\bar{S}$, with respect to the usual operations of addition and multiplication, is a ring. Constants do not belong to the ring $\bar{S}$. However, the product $a\bar{f}(p)$ belongs to $\bar{S}$ for every number a.

Denote by $\bar{J}_A$ the totality of all functions in $\bar{S}$ representable in the form $e^{-pA}\bar{g}(p)$, where $\bar{g}(p) \in \bar{S}$. $e^{-pA}\bar{g}(p)$ belongs to S for every function $\bar{g}(p) \in \bar{S}$. $\bar{J}_A$ is an ideal in the ring $\bar{S}$. Construct the set $\bar{S}_A = \bar{S}/\bar{J}_A$. The elements of the set $\bar{S}_A$ are the residue classes modulo $\bar{J}_A$. As is known, $\bar{S}_A$ will also be a ring. We shall denote the elements of the set $\bar{S}_A$ by the letters $\bar{F}_A, \bar{G}_A$, etc. Let $A = 1, 2, \ldots, n, \ldots.$ We obtain a sequence of sets $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_n, \ldots.$ Form the direct sum of these sets*. As is known, the elements of the direct sum will be systems $\bar{F} = \{\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_n, \ldots\}$, where $\bar{F}_n$ is an element of the set $\bar{S}_n$.

The direct sum is also a ring. If $\bar{G} = \{\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_n, \ldots\}$ is another element of the direct sum, then, by definition,

$$\bar{F} + \bar{G} = \{\bar{F}_1 + \bar{G}_1,\ \bar{F}_2 + \bar{G}_2, \dots, \bar{F}_n + \bar{G}_n, \dots\},$$

$$\bar{F} \cdot \bar{G} = \{\bar{F}_1 \cdot \bar{G}_1,\ \bar{F}_2 \cdot \bar{G}_2, \dots, \bar{F}_n \cdot \bar{G}_n, \dots\}, \qquad a\bar{F} = \{a\bar{F}_1,\ a\bar{F}_2, \dots,\ a\bar{F}_n, \dots\}.$$

* Sometimes it is more expedient to consider the direct sum of the rings $\bar{S}_A$ over all values of A from zero to infinity.

In the direct sum we single out the subset consisting of all elements

$$F = \{\overline{F}_1,\ \overline{F}_2, \dots, \overline{F}_n, \dots\},$$

for which $\overline{F}_1 \supset \overline{F}_2 \supset \cdots \supset \overline{F}_n \supset \overline{F}_{n+1} \supset \cdots$. Here the $\overline{F}_n$ are regarded as sets from $\overline{S}$. We shall denote the collection of all such elements by $\mathfrak{M}$. It is not difficult to prove that $\mathfrak{M}$ is a ring.

Define in the set L, alongside the usual notion of the sum of functions and the product of a function by a number, the product

$$f(t) * g(t) = \int_0^t f(t - \xi) g(\xi)\, d\xi, \qquad f \in L, \quad g \in L. \tag{2}$$

As is known, the product $f(t) * g(t)$ belongs to L, and with this definition of multiplication L will be a commutative ring.

Let $f(t) \in L$. Consider the integral

$$\overline{f}_n(p) = \int_0^n f(t) e^{-pt}\, dt. \tag{3}$$

Let $\overline{F}_n$ be the complex class containing the function $\overline{f}_n(p)$. Since

$$\overline{f}_{n+1}(p) - \overline{f}_n(p) = e^{-np} \int_0^1 f(n + \xi) e^{-p\xi}\, d\xi \in \overline{J}_n,$$

we have $\overline{F}_n \supset \overline{F}_{n+1}$. Therefore

$$F = \{\overline{F}_1, \overline{F}_2, \dots, \overline{F}_n, \dots\} \in \mathfrak{M}.$$

One can show that the mapping thus established is a one-to-one mapping of L onto $\mathfrak{M}$.

Theorem 1. The rings L and $\mathfrak{M}$ are isomorphic.

The proof of this theorem rests on the identity

$$\overline{h}_n(p) = \overline{f}_n(p)\,\overline{g}_n(p) - e^{-np}\int_0^n e^{-pt}\,dt \int_t^n f(t-\xi+n)g(\xi)\,d\xi.$$

Here $\overline{g}_n(p)$ is defined in the same way as $\overline{f}_n(p)$ (see (3)), and

$$\overline{h}_n(p) = \int_0^n h(t)e^{-pt}\,dt, \qquad \text{where} \qquad h(t) = \int_0^t f(t-\xi)g(\xi)\,d\xi.$$

This identity shows that $\overline{h}_n(p)$ is congruent to $\overline{f}_n(p)\overline{g}_n(p)$ modulo the ideal $\overline{J}_n$.

The ring L has no zero divisors. Extending it to its quotient field, we obtain the ring $R(L)$–the field of Mikusiński operators. It follows from Theorem 1 that $\mathfrak{M}$ has no zero divisors, and the extension $R(\mathfrak{M})$ will be isomorphic to the field of operators $R(L)$. The quotient field $R(\mathfrak{M})$ is not difficult to construct directly, and the apparatus of the theory of functions of a complex variable can be brought to bear on its study. In particular, by using the properties of the indicator for entire functions of completely regular growth, one can directly prove the absence of zero divisors in the ring $\mathfrak{M}$.

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References

1. J. Mikusinski, *Rachunek operatoròw*, Warszawa, 1953.

2. V. A. Ditkin, *Uspekhi Mat. Nauk*, **2**, no. 6 (1947).

Note: Figure translations are in progress. See original paper for figures.