MATHEMATICS

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. G. BOLTYANSKII

HOMOTOPIC CLASSIFICATION OF VECTOR FIELDS

(Presented by Academician P. S. Aleksandrov, 25 VI 1957)

As is known (^{1}), on a smooth closed manifold M^n one can construct a field of unit tangent vectors without singularities if and only if the Euler characteristic of the manifold M^n is equal to zero. In what follows all manifolds are assumed to be orientable, smooth, closed, and to have Euler characteristic equal to zero, and all vector fields are normalized (unit) and, unless otherwise stated, without singularities (i.e. continuous). Without loss of generality one may suppose that a Riemannian metric has been introduced on the manifold under consideration; in particular, this will make it possible to speak of the angles between vectors given at one and the same point.

In the present note a homotopic classification of vector fields on an n-dimensional manifold is given. For $n \leq 3$ it is obtained as a trivial consequence of certain known results (see below). For $n \geq 4$ the classification is obtained on the basis of my results on the homotopic classification of section surfaces and consists in the following: *to each element of the integral contrahomology group $H^{n-1}(M^n)$ (we replace the term "cogomology" by the term "contrahomology," which, in our opinion, is more correct) there correspond, for $n \geq 4$, exactly two homotopy classes of vector fields.*

More fully:

Theorem. Let K^n be a simplicial (or cellular) decomposition of an n-dimensional manifold M^n (on the triangulability of smooth manifolds see (^{2})). We shall call two vector fields on M^n $(n-1)$-homotopic if they, considered only on the $(n-1)$-dimensional skeleton K^{n-1}, are homotopic to each other. Then the $(n-1)$-homotopy classes of vector fields on M^n are in one-to-one correspondence with the elements of the group $H^{n-1}(M^n)$, and for $n \geq 4$ each $(n-1)$-homotopy class splits into exactly two homotopy classes, which gives the complete homotopic classification of vector fields on M^n for $n \geq 4$.

Proof. The set $T(M^n)$ of all unit tangent vectors to M^n is a skew product (the "tangent bundle") with base M^n and fiber S^{n-1}; the structure group is the group $SO(n)$ of rotations of the sphere S^{n-1}. Any two vector fields $\mathfrak{S}_1, \mathfrak{S}_2$ on M^n can, by means of a deformation, be brought into coincidence on the skeleton K^{n-2}, after which for them the $(n-1)$-dimensional difference $\partial^{n-1}(\mathfrak{S}_1, \mathfrak{S}_2)$

is defined, which is a contracycle, since both fields $\mathfrak{S}_1, \mathfrak{S}_2$ are given on the whole manifold M^n. The contracycle $\partial^{n-1}(\mathfrak{S}_1, \mathfrak{S}_2)$ itself depends on the choice of the indicated deformation, but its contrahomology class $D^{n-1}(\mathfrak{S}_1, \mathfrak{S}_2) \in H^{n-1}(K^n, \pi_{n-1}(S^{n-1}))$ is uniquely determined by the vector fields $\mathfrak{S}_1, \mathfrak{S}_2$. Fix some one vector field $\mathfrak{S}_0$. Then for any element $D^{n-1} \in H^{n-1}(K^n, \pi_{n-1}(S^{n-1}))$ there exists a vector field $\mathfrak{S}$ on M^n such that $D^{n-1}(\mathfrak{S}_0, \mathfrak{S}) = D^{n-1}$; furthermore, for $(n-1)$-homotopy of fields

$\mathfrak{S}_1$ and $\mathfrak{S}_2$ it is necessary and sufficient that the equality $D^{n-1}(\mathfrak{S}_0, \mathfrak{S}_1) = D^{n-1}(\mathfrak{S}_0, \mathfrak{S}_2)$ hold. Thus, the correspondence $\mathfrak{S} \to D^{n-1}(\mathfrak{S}_0, \mathfrak{S})$ establishes a one-to-one mapping of the set of all $(n-1)$-homotopy classes of vector fields onto the set of all elements of the group $H^{n-1}(K^n, \pi_{n-1}(S^{n-1}))$, or, what is the same thing, of the integral cohomology group $H^{n-1}(M^n)$. This is, in essence, the Hopf–Whitney classification theorem. It follows without difficulty from the usual properties of differences and obstructions (see, for example, ([3])).

It remains to show—and this is the content of the theorem—that each $(n-1)$-homotopy class splits, for $n \geq 4$, into exactly two homotopy classes. Here we shall use the classification theorem for secant surfaces mentioned in ([4]). All vector fields belonging to one and the same $(n-1)$-homotopy class can, by means of deformations, be made to coincide on the skeleton K^{n-1}. Therefore we shall consider only vector fields that coincide with one another on the skeleton K^{n-1}. According to the classification theorem mentioned in ([4]), the vector fields $\mathfrak{S}_1$ and $\mathfrak{S}_2$, coinciding on the skeleton K^{n-1}, are homotopic to one another if and only if

$$D^n(\mathfrak{S}_1, \mathfrak{S}_2) = \widetilde{Y}^2 \sim \Lambda^{n-2} + \mathrm{Sq}^2 \Lambda^{n-2} \qquad (n \geq 4), \tag{1}$$

where $D^n(\mathfrak{S}_1, \mathfrak{S}_2)$ is the class of the n-dimensional difference cohomology of these fields; the meaning of the expression standing on the right-hand side will be discussed below.

Main lemma. *For any element $\Lambda^{n-2} \in H^{n-2}(K^n, \pi_{n-1}(S^{n-1}))$, the expression standing on the right-hand side of relation (1) vanishes.*

From the main lemma and condition (1) it follows that two vector fields $\mathfrak{S}_1$ and $\mathfrak{S}_2$, coinciding on the skeleton K^{n-1}, are homotopic to one another if and only if

$$D^n(\mathfrak{S}_1, \mathfrak{S}_2) = 0. \tag{2}$$

But $D^n(\mathfrak{S}_1, \mathfrak{S}_2)$ is an element of the group $H^n(K^n, \pi_n(S^{n-1}))$, which is a group of order two (since $n \geq 4$). Therefore $D^n(\mathfrak{S}_1, \mathfrak{S}_2)$ can take two different values, whence from the necessary and sufficient condition for homotopy (2) there follows the existence of exactly two homotopy classes in each $(n-1)$-homotopy class of vector fields. Thus, it remains to prove the main lemma.

The difference cohomology class $\widetilde{Y}^2$ and the triples of groups needed for defining the operations $\sim$ and Sq^2 are described in detail in (^{3}) (pp. 112–115). In the case under consideration by us, of the tangent bundle $T(M^n)$, the difference cohomology class $\widetilde{Y}^2$ coincides with the two-dimensional characteristic Stiefel class of the manifold M^n. Since $\pi_{n-1}(S^{n-1})$ is a free cyclic group, and $\pi_n(S^{n-1})$ is a group of order two, we shall agree to identify the former with the additive group of integers, and the latter with the group of residues modulo 2. According to (^{3}), the operation Sq^2 is defined in the same way as in the Steenrod classification theorem $M^n \to S^{n-1}$, i.e. the pair of groups $(\pi_{n-1}(S^{n-1}), \pi_n(S^{n-1}))$ is obtained by considering ordinary multiplication of integers with subsequent reduction modulo 2. Finally, from formula (14) of the paper (^{3}) it follows easily (see, for example, pp. 194–196 of the paper (^{5}), where the computations are carried out in detail) that in defining the operation $\sim$ in formula (1) indicated above, ordinary multiplication of coefficients is also used (i.e. multiplication of residues modulo 2 by integers).

Let Λ^{n-2} be an arbitrary element of the group $H^{n-2}(K^n, \pi_{n-1}(S^{n-1}))$, and let $V^2 \in H_2(K^n, \pi_{n-1}(S^{n-1}))$ be the two-dimensional homology class Poincaré-dual to it. Since we have agreed to identify the group $\pi_{n-1}(S^{n-1})$ with the additive group of integers, V^2 is an integral

two-dimensional homology class. According to (^{6}), there exists in M^n a smooth two-dimensional orientable surface P^2 which, taken with a certain orientation, is a cycle determining the homology class V^2. Choose on P^2 one point a and construct on $P^2 \setminus a$ vector fields $v_1, \dots, v_{n-2}$, pairwise orthogonal to one another, orthogonal to the surface P^2, and tangent to M^n. In going around the point a along a small circle O lying on the surface P^2, the vector fields $v_1, \dots, v_{n-2}$ change, which is specified by a mapping of the circle O into $SO(n-2)$. It determines the "index" α of the point a, which is an element of the group $\pi_1(SO(n-2))$, i.e. an integer for $n = 4$ and a residue modulo 2 for $n > 4$.

Lemma 1. The relations $\mathrm{Sq}^2\Lambda^{n-2} = 0$ and $\alpha \equiv 0 \pmod 2$ are equivalent.

For the proof, denote by V_x, $x \in P^2$, the geodesic $(n-2)$-dimensional ball of radius ε, normal to P^2, with center at the point x. Choose the number $\varepsilon > 0$ so small that the balls do not intersect pairwise. The union of all these balls is a closed neighborhood U of the surface P^2 in M^n. Denote by φ_x, $x \neq a$, a homeomorphic mapping of the ball V_x onto a fixed $(n-2)$-dimensional ball E, carrying the directions of the vectors $v_1, \dots, v_{n-2}$, taken at the point x, into fixed pairwise orthogonal radii of the ball E. By φ denote the mapping of the set $U \setminus V_a$ into E which on the ball V_x, $x \neq a$, coincides with the mapping φ_x. Let now ψ be a mapping of the ball E onto S^{n-2}, carrying the whole boundary of the ball E into one point $q \in S^{n-2}$ and having degree 1. By f denote the mapping of the set $M^n \setminus V_a$ into S^{n-2} which coincides with $\psi \circ \varphi$ on $U \setminus V_a$ and carries the set $M^n \setminus U$ into the point q. For sufficiently small ε it may be assumed that the mapping f is defined on the entire skeleton K^{n-1}, and it is not difficult to see that its characteristic class $f^* s^{n-2}$ (where s^{n-2} is the generating element of the group $H^{n-2}(S^{n-2}))$ is equal to Λ^{n-2}. Therefore the second obstruction for the mapping f, according to Steenrod' s classification theorem (([5]), § 14), is equal to $\mathrm{Sq}^2\Lambda^{n-2}$. But this obstruction is determined by the mapping f considered on the $(n-1)$-dimensional sphere S^{n-1} lying in M^n and surrounding the ball V_a. This latter mapping can be investigated literally in the same way as in the works cited above (([3]), formula (14), or ([5]), pp. 194-196), from which we conclude that this mapping $S^{n-1} \to S^{n-2}$ determines an element of the group $\pi_{n-1}(S^{n-2})$ having the "same parity" as α. To prove Lemma 1 it remains to perform reduction modulo 2 in the case $n = 4$. (We note that in the proof of Lemma 1 we used other coefficient domains, but isomorphic to the former ones, i.e., for example, $\pi_{n-2}(S^{n-2})$ instead of $\pi_{n-1}(S^{n-1})$.)

Lemma 2. The congruence $\alpha \equiv 0 \pmod 2$ is equivalent to the fact that on the whole surface P^2 one can construct n mutually orthogonal vector fields (tangent to M^n, but arbitrarily positioned with respect to P^2).

For the proof, in addition to the fields $v_1, \dots, v_{n-2}$ normal to P^2, construct on $P^2 \setminus a$ two mutually orthogonal fields u_1, u_2 tangent to P^2. In going around the point a the fields u_1, u_2 make an even number of turns (since the surface P^2 has even Euler characteristic). We may assume that on one half of the circle O the fields u_1, u_2 are constant, while $v_1, \dots, v_{n-2}$ change, and on the other half of the circle the fields $v_1, \dots, v_{n-2}$ are constant, while u_1, u_2 change. Since the group $\pi_1(SO(n))$ has order two, the second of the indicated halves of the circle determines the trivial element of this group, while the first half determines an element which is trivial if and only if $\alpha \equiv 0 \pmod 2$. Thus the singularity of the vector fields $u_1, u_2, v_1, \dots, v_{n-2}$ in a neighborhood of the point a is removable if and only if $\alpha \equiv 0 \pmod 2$. (The removability of this singularity does not depend on any special choice of the fields, since it is connected with the characteristic class of the corresponding skew product.)

Lemma 3. On the surface $P^2 \subset M^n$, n mutually orthogonal fields can be constructed if and only if

$$\widetilde{Y}^2 \sim \Lambda^{n-2} = 0.$$

Indeed, the equality $\widetilde{Y}^2 \sim \Lambda^{n-2} = 0$ is equivalent to saying that the scalar product of the cycle P^2 and the contracycle $\tilde{y}^2 \in \widetilde{Y}^2$ is equal to zero, or, in other words, that the intersection index of the cycle P^2 with the $(n-2)$-dimensional Stiefel cycle of singularities is equal to zero. From this the validity of Lemma 3 follows without difficulty.

From Lemmas 1-3 it follows that

$$\mathrm{Sq}^2\,\Lambda^{n-2} = \widetilde{Y}^2 \sim \Lambda^{n-2}$$

(for both the right- and left-hand sides are residues modulo 2), whence the validity of the main lemma follows at once. Thus our main theorem is completely proved. (A particular, though completely trivial, case of this theorem, when $M^n = S^n$, was indicated without proof by Weier ([7]).)

Let us now turn to the cases $n \leqslant 3$. As is known, a manifold M^n is called **parallelizable** if on it one can construct a parallelizing system, i.e. n mutually orthogonal vector fields. The only (connected) one-dimensional manifold is the circle; it is trivially parallelizable. The only (connected) two-dimensional orientable manifold with zero Euler characteristic is the torus. It is also parallelizable. Finally, every orientable three-dimensional manifold is also parallelizable (^{8}). Thus, for $n \leqslant 3$, all the manifolds under consideration are parallelizable. But the homotopic classification of vector fields on a parallelizable manifold M^n is equivalent to the homotopic classification of mappings of this manifold into the sphere S^{n-1} (since each vector of the field can be decomposed with respect to the vectors of a parallelizing system). This classification is trivial for $n = 1, 2$, while the classification of mappings $M^3 \to S^2$ was given by L. S. Pontryagin (^{5}), §14.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
24 VI 1957

REFERENCES

[1] H. Hopf, Math. Ann., **96**, 225 (1927).
[2] S. Cairns, Ann. Math., **35**, 579 (1934).
[3] V. Boltyanskii, Izv. AN SSSR, ser. matem., **20**, 99 (1956).
[4] V. Boltyanskii, Tr. 3rd All-Union Math. Congress, **2**, 52, 1956.
[5] V. Boltyanskii, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **47** (1955).
[6] R. Thom, Comm. Math. Helv., **28**, 17 (1954).
[7] J. Weier, C. R., **243**, 1995 (1956).
[8] E. Stiefel, Comm. Math. Helv., **8**, 3 (1936).

Note: Figure translations are in progress. See original paper for figures.