STABILITY OF EQUILIBRIUM OF RODS FROM THE POINT OF VIEW OF THE MATHEMATICAL THEORY OF ELASTICITY

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Abstract

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K. F. VOITSEKHOVSKAYA

STABILITY OF EQUILIBRIUM OF RODS FROM THE POINT OF VIEW OF THE MATHEMATICAL THEORY OF ELASTICITY

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Until recently, problems on the stability of equilibrium of elastic bodies were solved by methods of the applied theory of elasticity or strength of materials, in which simplifying hypotheses were introduced concerning the character of the deformed or stressed states. V. V. Novozhilov ([2]) poses the general problem of the stability of an elastic body from the point of view of the mathematical theory of elasticity, without, however, considering solutions of concrete problems. A. Yu. Ishlinskii ([1]) develops the solution of a concrete problem: the stability of compression of an infinitely long strip under conditions of plane deformation, taking account of deformation of the boundaries. In the present article, by methods of the mathematical theory of elasticity, the spatial problem of the stability of compression of elastic circular rods is considered.

Let a cylinder of length l and radius R be compressed by a force p uniformly distributed over the end sections. The lateral surface is free of forces. Then the stressed state in cylindrical coordinates r, θ, z (the z-axis is directed along the axis of the cylinder; the r- and θ-axes are situated in the middle cross-section of the cylinder) is determined by the stresses

$$\sigma_r^0 = \sigma_\theta^0 = \tau_{r\theta}^0 = \tau_{\theta z}^0 = \tau_{zr}^0 = 0, \qquad \sigma_z^0 = -p, \tag{1}$$

to which correspond the displacements

$$u_r^0 = \frac{\nu p}{E}\, r, \qquad u_\theta^0 = 0, \qquad u_z^0 = -\frac{p}{E}\, z, \tag{2}$$

where E is Young' s modulus, ν is Poisson' s ratio.

We shall investigate whether, alongside the fundamental state of equilibrium of the cylinder, determined by formulas (1)–(2), there can exist some other state of equilibrium, infinitely close to the fundamental one, in which the lateral surface of the body is likewise free of forces, but may already be noncylindrical.

Let u_r, u_θ, and u_z be infinitely small additional displacements of the points of the cylinder from their positions in the initial deformed state, and let

$\sigma_r, \sigma_\theta, \sigma_z, \tau_{r\theta}, \tau_{\theta z}, \tau_{zr}$ be the corresponding infinitely small stresses associated with these displacements.

The relation between the displacements u_r, u_θ, u_z and the stresses $\sigma_r, \sigma_\theta, \sigma_z, \tau_{r\theta}, \tau_{\theta z}, \tau_{zr}$ is established by the relations of Hooke' s law

$$\begin{aligned} \sigma_r &= \lambda\Delta + 2\mu\frac{\partial u_r}{\partial r}, & \tau_{r\theta} &= \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \\ \sigma_\theta &= \lambda\Delta + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), & \tau_{\theta z} &= \mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right), \\ \sigma_z &= \lambda\Delta + 2\mu\frac{\partial u_z}{\partial z}, & \tau_{zr} &= \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \end{aligned} \tag{3}$$

where

$$\Delta = \frac{\partial u_r}{\partial r} - \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} - \frac{\partial u_z}{\partial z} \tag{4}$$

is the volume expansion of the material; λ and μ are Lamé constants.

The displacements u_r, u_θ, and u_z must satisfy the equilibrium equations

$$\begin{gathered} (\lambda+\mu)\frac{\partial \Delta}{\partial r} + \mu\nabla^2 u_r - \frac{\mu}{r^2}\left(u_r + 2\frac{\partial u_\theta}{\partial \theta}\right) = 0, \\ (\lambda+\mu)\frac{1}{r}\frac{\partial \Delta}{\partial \theta} + \mu\nabla^2 u_\theta + \frac{\mu}{r^2}\left(2\frac{\partial u_r}{\partial \theta} - u_\theta\right) = 0, \\ (\lambda+\mu)\frac{\partial \Delta}{\partial z} + \mu\nabla^2 u_z = 0, \end{gathered} \tag{5}$$

and the stresses $\sigma_r, \sigma_\theta, \sigma_z, \tau_{r\theta}, \tau_{\theta z}, \tau_{zr}$ must satisfy the boundary conditions:

$$\begin{gathered} (\sigma_r^0 + \sigma_r)\cos r\nu + (\tau_{r\theta}^0 + \tau_{r\theta})\cos\theta\nu + (\tau_{rz}^0 + \tau_{rz})\cos z\nu = 0, \\ (\tau_{r\theta}^0 + \tau_{r\theta})\cos r\nu + (\sigma_\theta^0 + \sigma_\theta)\cos\theta\nu + (\tau_{\theta z}^0 + \tau_{\theta z})\cos z\nu = 0, \\ (\tau_{zr}^0 + \tau_{zr})\cos r\nu + (\tau_{z\theta}^0 + \tau_{z\theta})\cos\theta\nu + (\sigma_z^0 + \sigma_z)\cos z\nu = 0 \end{gathered} \tag{6}$$

on the deformed lateral surface of the cylinder, with normal ν.

Using the relations

$$\cos r\nu : \cos\theta\nu : \cos z\nu = 1 : \left(-\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) : \left(-\frac{\partial u_r}{\partial z}\right)$$

and omitting products of small quantities, the boundary conditions (6) may be written as follows:

$$\sigma_r = 0, \qquad \tau_{r\theta} = 0, \qquad \tau_{rz} = -p\frac{\partial u_r}{\partial z} \quad \text{for } r = R + u_r^0 + u_r. \tag{7}$$

It is easy to see, by expanding the continuous functions $\sigma_r, \tau_{r\theta}$, and τ_{rz} in Taylor series, that, up to quantities of the second order of smallness, the boundary conditions (7) may be imposed not on the deformed surface $r = R + u_r^0 + u_r$, but on the original surface with rectilinear generator $r = R$, i.e.,

$$\sigma_r = 0, \qquad \tau_{r\theta} = 0, \qquad \tau_{rz} = -p\frac{\partial u_r}{\partial z} \quad \text{for } r = R. \tag{8}$$

Thus, to find the functions u_r, u_θ, and u_z, one must solve the system of differential equations (5) with boundary conditions (8).

Assuming that buckling takes place in one plane, say in the plane $\theta = 0$, we shall seek the functions u_r, u_θ, and u_z in the form

$$u_r = u(r)\cos\theta\cos az,$$

$$u_\theta = v(r)\sin\theta\cos az, \tag{9}$$

$$u_z = w(r)\cos\theta\sin az.$$

The parameter a determines the wavelength of the "disturbance" of the cylinder generator; obviously,

$$a = (2m+1)\frac{\pi}{l}, \tag{10}$$

where m is any integer, and $l/(2m+1)$ is the half-wavelength of the "disturbance" of the cylinder generator.

Substituting expressions (9) into the equilibrium equations (5), we obtain a system of three second-order differential equations with variable coefficients for determining the functions $u(r), v(r)$, and $w(r)$.

$$
\begin{gathered}
(\lambda+2\mu)\left(u''+\frac{1}{r}u'-\frac{1}{r^2}u\right)-\mu\left(a^2+\frac{1}{r^2}\right)u+\\
+(\lambda+\mu)\frac{1}{r}v'-\frac{\lambda+3\mu}{r^2}v+a^2(\lambda+\mu)w'=0,\\
\frac{\lambda+\mu}{r}u'+\frac{\lambda+3\mu}{r^2}u-\mu\left[v''+\frac{1}{r}v'-\left(a^2+\frac{1}{r^2}\right)v\right]+\\
+\frac{\lambda+2\mu}{r^2}v+\frac{(\lambda+\mu)a}{r}w=0,\\
(\lambda+\mu)a\left(u'+\frac{1}{r}u\right)+\frac{(\lambda+\mu)a}{r}v-\mu\left(w''+\frac{1}{r}w'-\frac{1}{r^2}w\right)+\\
+(\lambda+2\mu)a^2w=0.
\end{gathered}
\tag{11}
$$

We find the general solution of system (11) in the form

$$
\begin{gathered}
u(r)=-i\gamma I_0(iar)+\left(\frac{\alpha a+\gamma}{ar}+\beta r\right)I_1(iar)-i\gamma^*N_0(iar)+\\
+\left(\frac{\alpha^*a+\gamma^*}{ar}+\beta^*r\right)N_1(iar),\\
v(r)=-i\alpha I_0(iar)+\frac{\alpha a+\gamma}{ar}I_1(iar)-i\alpha^*aN_0(iar)+\\
+\frac{\alpha^*a+\gamma^*}{ar}N_1(iar),\\
w(r)=-i\beta rI_0(iar)-\left[\frac{\lambda+3\mu}{a(\lambda+\mu)}\beta-\gamma\right]I_1(iar)-\\
-i\beta^*rN_0(iar)-\left[\frac{\lambda+3\mu}{a(\lambda+\mu)}\beta^*-\gamma^*\right]N_1(iar),
\end{gathered}
\tag{12}
$$

where $I_0(iar)$, $I_1(iar)$ and $N_0(iar)$, $N_1(iar)$ are, respectively, Bessel and Neumann functions of zero and first order with purely imaginary argument iar; α, β, γ, α^*, β^*, γ^* are constants of integration. In the case of a solid cylinder the functions u, v, and w must be bounded as $r\to 0$; therefore in the solution (12) one must set

$$
\alpha^*=\beta^*=\gamma^*=0. \tag{13}
$$

Thus, according to formulas (9) and (12), taking into account the equalities (13), we have

$$
\begin{aligned}
u_r(r,\theta,z) &= \left[-i\gamma I_0(iar) + \left(\frac{\alpha a+\gamma}{ar} + \beta r\right) I_1(iar)\right]\cos\theta\cos az,\\
u_\theta(r,\theta,z) &= \left[-i\alpha I_0(iar) + \frac{\alpha a+\gamma}{ar} I_1(iar)\right]\sin\theta\cos az,\\
u_z(r,\theta,z) &= \left\{-i\beta r I_0(iar) - \left[\frac{\lambda+3\mu}{a(\lambda+\mu)}\beta - \gamma\right] I_1(iar)\right\}\cos\theta\sin az.
\end{aligned}
\tag{14}
$$

Subjecting the solution (14) to the boundary conditions (8) and using formulas (3) and (4), we find three linear homogeneous equations with respect to the unknowns α, β, and γ:

$$
\begin{aligned}
&\alpha(tI_0 - 2I_1) + \beta R^2\left(tI_0 - \frac{\lambda}{\lambda+\mu} I_1\right) + \frac{\gamma}{a}\left[(t^2-2)I_1 + tI_0\right] = 0,\\
&\alpha\left[(t^2-4)I_1 + 2tI_0\right] - \beta R^2 I_1 + \frac{2\gamma}{a}(tI_0 - 2I_1) = 0,\\
&-\alpha a^2(p+\mu)I_1 + \beta\left\{(p+2\mu)t^2 I_1 - \mu\left[\frac{2(\lambda+2\mu)}{\lambda+\mu} tI_0 - \frac{\lambda+3\mu}{\lambda+\mu} I_1\right]\right\} +\\
&+\gamma a(p+2\mu)(tI_0 - I_1) = 0,
\end{aligned}
\tag{15}
$$

where, for brevity of notation, we have denoted

$$
I_0 \equiv I_0(t), \qquad I_1 \equiv I_1(t), \qquad t = iaR. \tag{16}
$$

The system (15) has a nontrivial solution only for the following value of the compressive force:

$$
p_i = 2\mu\left\{-1 + \frac{I_1\left[\frac{4\lambda+7\mu}{\lambda+\mu} t^2 I_0^2 + \left(\frac{\lambda+2\mu}{\lambda+\mu} t^2 - 2\frac{5\lambda+9\mu}{\lambda+\mu}\right) tI_0 I_1 + \left(4\frac{\lambda+2\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} t^2\right) I_1^2\right]}{t\left[2t^2 I_0^3 + \left(t^2 - \frac{5\lambda+3\mu}{\lambda+\mu}\right) tI_0^2 I_1 + \left(\frac{\lambda+2\mu}{\lambda+\mu} t^2 + 2\frac{\lambda-\mu}{\lambda+\mu}\right) I_0 I_1^2 + \left(t^2 - \frac{4\lambda+5\mu}{\lambda+\mu}\right) tI_1^3\right]}\right\},
$$

as is not difficult to verify by setting the determinant of the system (15) equal to zero.

The right-hand side of the last equality can be represented as an expansion in powers of the parameter t.

$$
p_i = -\frac{\mu(3\lambda+2\mu)}{4(\lambda+\mu)} t^2\left\{1 - \frac{4\lambda^2+\lambda\mu-\mu^2}{6(\lambda+\mu)(3\lambda+2\mu)} t^2 - \frac{49\lambda^3+830\lambda^2\mu+975\lambda\mu^2+322\mu^3}{1152(\lambda+\mu)^2(3\lambda+2\mu)} t^4 - \cdots\right\}. \tag{17}
$$

Formula (17) can be compared with Euler' s formula

$$P_{\mathrm{E}} = \frac{(2m+1)^2\pi^2 EI}{l^2} \qquad (m = 0, 1, \ldots) \tag{18}$$

for the longitudinal bending of a circular rod of length l with hinged ends. For this, one should replace the moment of inertia I by its value $\pi R^4/4$, the force P_{E} by the force $p_{\mathrm{E}}\pi R^2$, referred to unit area, the modulus of elasticity E by the expression $\mu\dfrac{3\lambda+2\mu}{\lambda+\mu}$, and use the notation $t = iaR$; then formula (18) takes the form

$$p_{\mathrm{E}} = -\frac{\mu(3\lambda+2\mu)}{4(\lambda+\mu)}\,t^2. \tag{19}$$

Thus formula (19), obtained by methods of strength of materials, is the limiting case as $t \to 0$ of formula (17), derived on the basis of the general equations of the mathematical theory of elasticity.

The difference δ between the minimum values ($m = 0$) of the critical forces p_i and p_{E} (Table 1), computed by formulas (18) and (19) for $\lambda = 1.2 \cdot 10^6$ kg/cm^2, $\mu = 0.8 \cdot 10^6$ kg/cm^2, is of the order of fractions of a percent and, as the flexibility $2l/R$ decreases from 628 to 62.8, increases from 0.001 to 0.1%.

Table 1

t	$0.01i$	$0.03i$	$0.05i$	$0.07i$	$0.09i$	$0.1i$
P_i (kg/cm^2)	52.0005	468.0410	1300.3160	2549.2115	4215.3017	5205.0243
P_{E} (kg/cm^2)	52	468	1300	2548	4212	5200
δ (%)	0.001	0.009	0.024	0.047	0.078	0.097

In conclusion, I consider it my pleasant duty to express my deep gratitude to Academician of the Academy of Sciences of the Ukrainian SSR A. Yu. Ishlinskii for posing the problem and supervising its solution.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

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REFERENCES

1. A. Yu. Ishlinskii, *Ukr. Mat. Zh.*, 6, No. 2, 140 (1954).

2. V. V. Novozhilov, *Foundations of the Nonlinear Theory of Elasticity*, L.–M., 1948.

Note: Figure translations are in progress. See original paper for figures.