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Abstract

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ON A BOUNDARY VALUE PROBLEM FOR LINEAR ELLIPTIC SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS IN THE PLANE

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Let us consider the system of differential equations

$$\begin{aligned} a_{11}u_x + a_{12}u_y + b_{11}v_x + b_{12}v_y + c_{11}u + c_{12}v = f_1; \\ a_{21}u_x + a_{22}u_y + b_{21}v_x + b_{22}v_y + c_{21}u + c_{22}v = f_2. \end{aligned} \tag{1}$$

Here $a_{ij}(x,y)$, $b_{ij}(x,y)$ are bounded measurable functions satisfying the condition of uniform ellipticity

$$4\begin{vmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{vmatrix} \cdot \begin{vmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{vmatrix} - \left\{ \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} a_{12} & b_{11} \\ a_{22} & b_{21} \end{vmatrix} \right\}^2 \geq k > 0,$$

$$\begin{vmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{vmatrix} : \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0, \qquad \begin{vmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{vmatrix} : \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0 \tag{2}$$

in some simply connected, finite domain D; $c_{ij}(x,y)$, $f_i(x,y)$ belong to $L_p(D)$ for $p > 2$.

For the system (1) we shall solve the boundary value problem

$$\alpha(t)u(t) + \beta(t)v(t)\big|_{\Gamma} = 0, \tag{3}$$

where Γ is the boundary of the domain D; $\alpha(t)$, $\beta(t) \in \mathrm{Lip}(\nu, \Gamma)$, $0 < \nu < 1$, are functions prescribed on Γ, Hölder continuous, and $\alpha^2 + \beta^2 = 1$.

Introducing the notation

$$z = x + iy, \qquad w = u + iv, \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

we can reduce the problem (1)–(3) to the form

$$\frac{\partial w}{\partial \bar{z}} + \mu_1(z)\frac{\partial w}{\partial z} + \mu_2(z)\frac{\partial \overline{w}}{\partial \bar{z}} + a(z)w(z) + b(z)\overline{w(z)} = g(z), \tag{4}$$

$$\operatorname{Re}\{(\alpha - i\beta)w\}\big|_\Gamma = 0. \tag{5}$$

It follows from (2) that

$$|\mu_1(z)| + |\mu_2(z)| \leq q < 1. \tag{6}$$

We shall seek the solution in the generalized sense of S. L. Sobolev, i.e. $w(z) \in W_p^{(1)}(D)$, $p > 2$ (^{1}).

For the system $\partial w/\partial\bar{z} + aw + b\bar{w} = f$ this problem was solved by I. N. Vekua (^{2}). The system (1) can be reduced to the indicated system in the case when $\mu_1(z)$ and $\mu_2(z)$ are differentiable (^{3}). In the case when $u(t)$ (or $v(t)$) is prescribed on Γ, this problem was considered by Bers and Nirenberg (^{4}).

We shall assume that Γ is a Jordan curve with continuous curvature. Then, with the aid of a conformal mapping $\zeta = f(z)$, we can reduce the problem to the case when D is the disk $|z| \leq 1$, with $0 < c_1 < |f'(z)| < c_2 < \infty$ (^{5,6}) uniformly in D.

Let us call the integer

$$^{(2)}\ n = \frac{1}{2\pi}\{\arg(\alpha + i\beta)\}_\Gamma$$

the **index of the problem**; then the boundary condition can be rewritten in the form

$$\operatorname{Re}\{t^{-n}e^{p(t)}w(t)\}_\Gamma = 0, \tag{7}$$

where

$$p(z) = q_1(z) + iq_2(z) = \frac{1}{2\pi}\int_\Gamma q_2(t)\frac{t+z}{t-z}\,ds, \tag{8}$$

$$q_2(t)\big|_\Gamma = \arg[\alpha(t) + i\beta(t)] - n\arg t \in \operatorname{Lip}(\nu, \Gamma), \qquad 0 < \nu < 1.$$

If z approaches the boundary point t, then $p'(z)$ satisfies the condition $|p'(z)| \leq K|z-t|^{\nu-1}$ (7). Making the substitution $w_1(z) = e^{p(z)}w(z)$, we bring the system and the boundary condition to the final form (for the new function and coefficients we keep the old notation)

$$\frac{\partial w}{\partial \bar z} + \mu_1(z)\frac{\partial w}{\partial z} + \mu_2(z)\frac{\partial \bar w}{\partial z} + a(z)w + b(z)\bar w = g(z), \tag{9}$$

$$\operatorname{Re}\{t^{-n}w(t)\}_\Gamma = 0. \tag{10}$$

I. The case of nonnegative index, $n \geq 0$.

From the results of (8,9) it follows that any function $w(z) \in W_p^{(1)}(D)$, $p > 2$, satisfying condition (10) for $n \geq 0$, can be represented in the form

$$w(z) = -\frac{1}{\pi}\iint_D \left[\frac{\rho(\zeta)}{\zeta - z} + \frac{z^{2n+1}\overline{\rho(\zeta)}}{1 - z\bar\zeta}\right] dT_\zeta +$$

$$+a_0 + \cdots + a^{n-1}z^{n-1} + icz^n - \bar a_{n-1}z^{n+1} \cdots - \bar a_0 z^{2n} = T_1\rho + \psi(z); \tag{11}$$

$\rho(\zeta) \in L_p(D)$; $a_0, \dots, a_{n-1}$ are arbitrary complex numbers; c is an arbitrary real number. We shall regard $L_p(D)$ and $W_p^{(1)}(D)$ as spaces of complex functions over the field of real numbers.

Following the method proposed by I. N. Vekua (10), we substitute the expression for $w(z)$ (11) into equation (9); then we obtain the singular integral equation

$$\rho + \mu_1 S_1\rho + \mu_2 \bar S_1\rho + aT_1\rho + b\bar T_1\rho = g - \mu_1\psi'(z) - \mu_2\overline{\psi'(z)} - a\psi(z) - b\bar\psi(z), \tag{12}$$

where

$$S_1\rho = \frac{\partial}{\partial z}T_1\rho = -\frac{1}{\pi}\iint_D \left[\frac{\rho(\zeta)}{(\zeta - z)^2} + \frac{z^{2n+1}\bar\zeta}{(1 - z\bar\zeta)^2}\overline{\rho(\zeta)} + (2n+1)z^{2n}\frac{\overline{\rho(\zeta)}}{1 - z\bar\zeta}\right] dT_\zeta.$$

The singular operator

$$S\rho = -\frac{1}{\pi}\iint_D \left[\frac{\rho(\zeta)}{(\zeta - z)^2} + \frac{z^{2n+1}\bar\zeta}{(1 - z\bar\zeta)^2}\overline{\rho(\zeta)}\right] dT_\zeta$$

is a bounded linear operator in $L_p(D)$, $1 < p < \infty$. $\|S\rho\|_{L_p} \leq A_p\|\rho\|_{L_p}$, A_p depends continuously on p. It may be calculated that $A_2 = 1$. Therefore we can choose the number p from the interval $2 < p < 2 + \varepsilon$ so that $qA_p < 1$. Then the operator $\rho + \mu_1 S\rho + \mu_2 \bar S\rho$ has an inverse; hence equation (12) is equivalent to a Fredholm equation.

We shall prove that the homogeneous equation (12) ($g(z) = \psi(z) = 0$) has no solutions different from zero. Let $\rho(z)$ be such a solution. Then $w(z) = T_1\rho$ satisfies the homogeneous equation (9) ($g(z) = 0$) and, by the known theorem on the representation of solutions (4, 13–15),

$$w(z) = e^{S_0(z)} f[\chi(z)], \tag{13}$$

where $\chi(z)$ is a homeomorphism of the disk $|z| < 1$ onto itself; $f(\zeta)$ is a function analytic in the disk $|\zeta| < 1$; $S_0(z)$, $\chi(z)$, $\psi(\zeta) = \chi^{-1}(\zeta) \in W_p^{(1)}(D)$, $p > 2$, and therefore satisfy the Hölder condition with exponent $\frac{p-2}{p}$ (10), $\operatorname{Im} S_0(z)\big|_\Gamma = 0$.

It follows from (3) that the mapping $\zeta = \chi(z)$ preserves orientation. Substituting (13) into (10), we obtain a boundary-value problem for the analytic function $f(\zeta)$

$$\operatorname{Re}\left\{\frac{e^{S_0(z)} f[\chi(z)]}{z^n}\right\}\bigg|_\Gamma = 0. \tag{14}$$

Solving this problem, we obtain for $w(z)$ the expression

$$\begin{aligned} w(z) = e^{S_0(z)+np[\chi(z)]}\bigl[a_0 + a_1\chi(z) + \dots + a_{n-1}\chi^{n-1}(z) +{} \\ + ic\chi^n(z) - \dots - \bar{a}_0\chi^{2n}(z)\bigr]; \end{aligned} \tag{15}$$

$\rho(\zeta)$ is a function analytic in D.

In view of (10), $z^{-n}w(z)$ is a purely imaginary function; therefore on Γ it can be represented as the difference $\Phi(z) - \overline{\Phi}(z)$, where $\Phi(z)$ is analytic in the disk $|z| \leqslant 1$. From the expression for $T_1\rho$ it follows that $\int_L z^{-k}w(z)\,ds = 0$, $k = 0, \dots, 2n$. Hence

$$w(z)\big|_\Gamma = \operatorname{Im}\{z^{n+1}\Phi_1(z)\}\big|_\Gamma .$$

$\Phi_1(z)$ is analytic in D, but then $w(z)$ has on Γ at least $2(n+1)$ zeros. Therefore it follows from (15) that $w(z) = 0$ and $\rho(z) = \partial w/\partial\bar{z} = 0$.

Thus, we have proved the following theorem:

Theorem 1. *In the case of a nonnegative index, the nonhomogeneous boundary-value problem (9), (10) is always solvable, and the corresponding homogeneous problem has* $2n + 1$ *linearly independent solutions.*

II. **The case of negative index,** $n < 0$. Denote $m = -n$. As in the case $n \geqslant 0$, from $(8, 9)$ there follows the representation for functions satisfying condition (10):

$$w(z) = -\frac{1}{\pi}\iint_D \left[\frac{\rho(\zeta)}{\zeta - z} + \frac{\bar{\zeta}^{2m-1}\rho(\bar{\zeta})}{1 - z\bar{\zeta}}\right] dT_\zeta = T_2\rho, \tag{16}$$

where $\rho(\zeta) \in L_p(D)$ and satisfies $2m-1$ conditions

$$f_l(\rho) = \mathrm{Re}\left\{\iint_D \left[\zeta^{m-l-1}\rho(\zeta) + \bar{\zeta}^{m+l-1}\rho(\bar{\zeta})\right] \, dT_\zeta\right\} = 0,$$

$$f_0(\rho) = \mathrm{Re}\left\{\iint_D \zeta^{m-1}\rho(\zeta)\, dT_\zeta\right\} = 0, \tag{17}$$

$$f_{-l}(\rho) = \mathrm{Im}\left\{\iint_D \left[\zeta^{m-l-1}\rho(\zeta) + \bar{\zeta}^{m+l-1}\rho(\bar{\zeta})\right] \, dT_\zeta\right\} = 0, \quad l = 1, \dots, m-1.$$

The functions $\rho(\zeta)$ satisfying conditions (17) form a subspace $L_{p,2m-1}(D)$ of the space $L_p(D)$, which has index $2m-1$. Therefore any element of $L_p(D)$ can be represented in the form

$$\rho(\zeta) = \rho_{2m-1} + \sum_{-(m-1)}^{m-1} \lambda_k \tau_k,$$

where $\rho_{2m-1} \in L_{p,2m-1}(D)$; the τ_k are linearly independent and $\bar{\tau}_k \in L_{p,2m-1}(D)$. As τ_k one may take

$$\tau_l = \bar{z}^{m-l-1}, \quad \tau_0 = \bar{z}^{m-1}, \quad \tau_{-l} = i\bar{z}^{m-l-1}, \quad l = 1, \dots, m-1.$$

Substituting (16) into (9), we obtain the singular integral equation

$$\rho + \mu_1 S_2\rho + \mu_2\overline{S_2\rho} + aT_2\rho + b\overline{T_2\rho} = g,$$

$$S_2\rho = \frac{\partial T_2\rho}{\partial z} = -\frac{1}{\pi}\iint_D \left[\frac{\rho(\zeta)}{(\zeta - z)^2} + \frac{\bar{\zeta}^{2m}\rho(\bar{\zeta})}{(1 - z\bar{\zeta})^2}\right] dT_\zeta. \tag{18}$$

A direct calculation shows that $\|S_2\|_{L_2} = 1$. Therefore equation (18) is equivalent to a Fredholm equation. Let us show that the corresponding homogeneous equation has no solutions different from zero. Let $\rho(z)$ be such a solution,

$$\rho(z) = \rho_{2m-1} + \sum_{-(m-1)}^{m-1} \lambda_k \tau_k,$$

$$w(z) = T_2\rho = T_2\rho_{2m-1} + i\sum_{1}^{m-1} \lambda_{-k}\frac{z^{m-k}}{m-k} + \sum_{0}^{m-1} \lambda_k \frac{\bar{z}^{m-k}}{m-k}.$$

Using, as before, the theorem on the representation of solutions (4,[13–15]), we shall have

$$\operatorname{Re}\{t^m w(t)\}\big|_\Gamma = \operatorname{Re}\{t^m e^{S_0(z)} f[\chi(z)]\}\big|_\Gamma = \operatorname{Re}\left\{\frac{\lambda_0}{m} + \sum_{k=1}^{m-1} \frac{\lambda_k + i\lambda_{-k}}{m-k} z^k\right\}. \quad (19)$$

Considering the inverse homeomorphism $\psi(\zeta) = \chi^{-1}(\zeta)$ $(\psi(\zeta)|_\Gamma = \zeta e^{i\theta(\zeta)})$, we may write equality (19) in the form

$$\left[e^{S_0[\psi(\zeta)]-mq(\zeta)} \operatorname{Re}\{\zeta^m e^{mp(\zeta)} f(\zeta)\}\right]\big|_\Gamma = \operatorname{Re}\left\{\frac{\lambda_0}{m} + \sum_{1}^{m-1} \frac{\lambda_k + i\lambda_{-k}}{m-k} z^k\right\}\Bigg|_\Gamma;$$

$p(\zeta)$ is a function analytic in D, with $\operatorname{Im} p(\zeta)|_\Gamma = \theta(\zeta)$, $q(\zeta) = \operatorname{Re} p(\zeta)$. From the fact that $\operatorname{Re}\{\zeta^m e^{mp(\zeta)} f(\zeta)\}$ has on Γ not fewer than $2m$ zeros, it follows that $\lambda_k = 0$, $k = 0, \pm 1, \dots, \pm(m-1)$. Since $m > 0$, $f(\zeta) = 0$. Consequently, $\rho = \partial w/\partial \bar{z} = 0$. Thus the theorem has been proved:

Theorem 2. *In the case of a negative index, the homogeneous problem has only the zero solution, and for the solvability of the nonhomogeneous problem it is necessary and sufficient that*

$$\rho(z) = [I + \mu_1 S_2 + \mu_2 \overline{S}_2 + aT_2 + b\overline{T}_2]g$$

satisfy conditions (17).

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Note: Figure translations are in progress. See original paper for figures.

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