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Abstract

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ON FLUCTUATIONS IN QUARTZ OSCILLATORS

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The question of the fluctuation limit of the frequency stability of quartz oscillators has not yet been resolved. In the present work we set forth a method and the results of considering fluctuations in quartz oscillators used in practice. In doing so, symbolic differential equations are solved using the method of the small parameter ([1]). In problems on fluctuations in self-oscillatory systems, an essential role is played only by the part of the fluctuation spectrum that lies within the band of the system ([1,2]).

A peculiarity of oscillators with quartz is that they have two substantially different bands: the first, determined by the time constant of the circuit regenerated by the tube, and the second, determined by the time constant of the quartz resonator under the action of the tube and the circuit. The second band is considerably narrower than the first. Therefore the fluctuations of the amplitude of the quartz oscillations have a spectrum much narrower than the fluctuations of the amplitude of the circuit.

When quartz oscillators are considered by the method of the small parameter, allowance for the peculiarities of the quartz resonator leads to nonsymmetry of the orders of smallness of the right-hand sides of the equations of the system ([3]). When fluctuations in such systems are considered, the peculiarities of the quartz resonator lead to a difference in the orders of smallness of the widths of the spectra of the amplitude fluctuations of the quartz oscillations and of the circuit.

Therefore, in order to obtain a stationary process in a self-oscillatory system with quartz described by the equations:

$$\begin{aligned} \frac{d^2x}{dt^2} + x &= \mu f\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}, \mu\right) + \mu^2 F_1(t), \\ \frac{d^2y}{dt^2} + y &= \mu^2 g\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}, \mu\right) + \mu^3 F_2(t), \end{aligned} \tag{1}$$

where $F_1(t)$ and $F_2(t)$ are stationary random functions, it is necessary to regard the time derivative t of the amplitude of the quartz oscillation as a quantity of

Fig. 1

Figure 1: Fig. 1

a higher order of smallness relative to μ than the derivative of the amplitude of the circuit oscillation.

If the solution of (1) is sought in the form

$$\begin{aligned} x &= P\cos(t-\varphi) + Q\sin(t-\varphi) + \mu \text{ (harmonics)}, \\ y &= R\cos(t-\varphi) + \mu^2 \text{ (harmonics)}, \end{aligned} \tag{2}$$

then, in order to obtain stationary solutions, it is necessary to assume that

$$\frac{dP}{dt} \sim \frac{dQ}{dt} \sim \mu, \qquad \frac{dR}{dt} \sim \mu^2. \tag{3}$$

Relations (3) are obtained formally if P and Q are regarded as functions of $\tau = \mu t$, and R as a function of $\tau_1 = \mu^2 t$. In this case

$$\frac{dP}{dt} = \mu\dot{P}, \qquad \frac{dQ}{dt} = \mu\dot{Q}, \qquad \frac{dR}{dt} = \mu^2 R'.$$

Here and below a dot denotes differentiation with respect to τ, and a prime with respect to τ_1.

Because of the coupling between the circuit and the quartz, the equations for fluctuations in the circuit will contain terms determined by fluctuations in the quartz, and conversely. The differences in the time constants make it possible to assume that the circuit follows the quartz without inertia, while for the amplitude of the quartz only so narrow a low-frequency portion of the fluctuation spectrum in the circuit is important that, over it, the spectral density of these fluctuations may be regarded as constant.

Fig. 1

The analysis was carried out for two typical oscillator circuits and for a pulling circuit. We shall restrict ourselves here to the oscillator circuit with quartz in the grid circuit, shown in simplified form in Fig. 1. The dynamical equations of this circuit were compiled and analyzed in work [3]. Taking account of shot fluctuations of the anode current and thermal noises reduces to the fact that fluctuation forces must be added to the right-hand sides of the equations obtained in [3] (the notation is clear from Fig. 1 and from [3]):

$$\mu^2 F = \frac{1}{\sqrt{\omega}\,(C_1 + C)}\left(i \;+ \frac{E_1}{\omega L_1}\right), \qquad \mu^3 G = \frac{E_2}{\sqrt{\omega^2 L_2\,(C_s + C)}}. \tag{4}$$

Carrying out the usual procedure, we obtain, in the first approximation, the equations for the fluctuations:

$$2\dot{\xi}_1 = -\theta_1\xi_1 + \Delta\eta_1 + \sigma\rho_1\left(1 - \frac{3R_0^2}{4}\right) + F_\perp,$$

$$2\dot{\eta}_1 = -\Delta\xi_1 - \theta_1\eta_1 - \chi_1\rho_1 + F_\parallel, \tag{5}$$

$$2\rho_1' = \chi_2(\Delta+\delta)\eta_1 - \chi_2\theta_1\xi_1 - \theta_2\rho_1 - \chi_2\sigma\left(1 - \frac{3R_0^2}{4}\right)\rho_1 + G_\perp + \chi_2 F_\perp,$$

$$\chi_1 = 0,$$

where ξ_1, η_1, ρ_1, and χ_1 are, respectively, the fluctuations of the amplitude of the circuit and of the quartz, and the phase fluctuations.

As is seen from (5), the phase of the quartz does not fluctuate in the first approximation. The equation for the fluctuations of the phase is found in the second approximation and has the form

$$2R_0\chi_2' = -\chi_2(\Delta+\delta)\xi_1 - \chi_2\theta_1\eta_1 - (\chi_1\chi_2 + 2\delta_1)\rho_1 + G_\parallel + \chi_2 F_\parallel. \tag{6}$$

The coefficients of these equations are related to the parameters of the circuit of Fig. 1 in the same way as in work [3]. $F_\parallel$, $F_\perp$, $G_\parallel$, and $G_\perp$ are, respectively, the "parallel" and "perpendicular" components of the shot and thermal fluctuations. Their correlation functions have the form [1,2]:

$$\overline{F_\parallel(\tau)F_\parallel(\tau')} = \overline{F_\perp(\tau)F_\perp(\tau')} = 2\mu C\delta(\tau' - \tau),$$

$$\overline{G_\parallel(\tau)G_\parallel(\tau')} = \overline{G_\perp(\tau)G_\perp(\tau')} = 2\mu D\delta(\tau' - \tau).$$

After the corresponding calculations, for the correlation function of the ampli-

of the quartz shot fluctuations, we have

$$\overline{\rho_1(\tau_1)\rho_1(\tau_1 + \mu\theta)} = \frac{\mu^2}{4\gamma}\left[D + \varkappa_2^2 C\frac{\delta^2}{\Delta^2 + \theta_1^2}\right]e^{-\mu\gamma|\theta|}. \tag{7}$$

Here $1/\gamma$ represents the effective Q-factor of the quartz in the circuit. Thus, the influence of the shot effect on the amplitude fluctuations of the quartz is weakened not only by the coupling $\varkappa_2$, but also by the factor $\dfrac{\delta^2}{\Delta^2+\theta_1^2}$, which is the smaller, the higher the Q-factor of the quartz.

For the mean square of the phase run-up we obtain

$$\overline{\chi_{2\tau}^2} = \frac{\mu}{2R_0^2}\left[D+\varkappa_2^2 C\right]\tau - \frac{\mu C\varkappa_2^2}{8R_0^2\theta_1}\left\{\frac{|(\Delta+\delta)+i\theta_1|^2}{p_1^2}\left[e^{p_1\tau}-1-p_1\tau\right]+\frac{|(\Delta+\delta)-i\theta_1|^2}{p_2^2}\left[e^{p_2\tau}-1-p_2\tau\right]\right\} + \frac{\quad}{8R_0^2\gamma}$$

$$\times\left[\frac{2\delta_1}{\delta}(\Delta^2+\theta_1^2)-\theta_1\sigma\left(1-\frac{3R_0^2}{4}\right)-\Delta\varkappa_1\right]^2\left[e^{-\mu\gamma\tau}-1+\mu\gamma\tau\right]. \qquad (8)$$

Here p_1 and p_2 are the roots of the characteristic equation

$$p^2+\theta_1 p+\frac{\Delta^2+\theta_1^2}{4}=0.$$

The first term gives the initial diffusion of the phase, caused by the direct action of the fluctuating forces. The second term takes into account the influence of the amplitude fluctuations of the quartz on the phase of the quartz. The third term describes the influence of the amplitude fluctuations of the quartz on the phase of the quartz.

Since $\mu\gamma \ll |\operatorname{Re} p_1| = \theta_1/2$, it is necessary to distinguish three time intervals:

1) $\tau\theta_1 \ll 1$ (initial behavior of the mean square of the phase run-up)

$$\overline{\chi_{2\tau}^2} = \frac{\mu}{2R^2}\left[D+\varkappa_2^2 C\right]\tau = D_{\mathrm{n}}\tau.$$

2) $\dfrac{1}{\theta_1} \ll \tau \ll \dfrac{1}{\mu\gamma}$ (intermediate interval)

$$\overline{\chi_{2\tau}^2} = \frac{\mu}{2R_0^2}\left[D+\varkappa_2^2\frac{\delta^2}{\Delta^2+\theta_1^2}C\right]\tau = D_{\mathrm{p}}\tau.$$

Thus, after a time $\dfrac{1}{\theta_1}$ has elapsed, the action of the shot effect is weakened by the factor $\dfrac{\delta^2}{\Delta^2+\theta_1^2}$. This is a consequence of the correlation of the amplitude fluctuations of the circuit and the phase fluctuations.

3) $\tau \gg \dfrac{1}{\mu\gamma}$ (steady-state diffusion)

Fig. 2

Figure 2: Fig. 2

$$\overline{\chi_{2\tau}^2} = \frac{\mu}{2R_0^2}\left[D + \varkappa_2^2 \frac{\delta^2}{\Delta^2+\theta_1^2} C\right] \times$$

$$\left\{1 + \frac{\varkappa_2^2\delta^2}{4\gamma^2(\Delta^2+\theta_1^2)}\left[\frac{2\delta_1}{\delta}(\Delta^2+\theta_1^2) - \theta_1\delta\left(1 - \frac{3R_0^2}{4}\right) - \Delta\varkappa_1\right]^2\right\}\tau = D_\text{y}\tau.$$

Thus, the amplitude fluctuations of the quartz somewhat accelerate the growth of the phase in comparison with the intermediate interval. The behavior of the mean square of the phase run-up of the quartz in time has the form shown in Fig. 2.

The calculation for the specific circuit, taken from (3), gave:

$$\mu^5 D_\text{n} \approx 2\cdot 10^{-14}, \qquad \mu^5 D_\text{p} \approx 4\cdot 10^{-3}\cdot \mu^5 D_\text{n} \approx 8\cdot 10^{-17},$$

$$\mu^5 D_\text{y} \approx 10^{-2}\mu^5 D_\text{n} \approx 2\cdot 10^{-16}, \qquad \frac{1}{\mu\theta_1} \approx 20, \qquad \frac{1}{\mu^2\gamma} \approx 250.$$

It follows from this that over times $\sim \frac{1}{\mu^2\gamma}$ the phase has time to increase only to very small fractions of unity (10^{-10}). Therefore the initial behavior of the mean square of the phase run-up has almost no effect on the spectral line, and, consequently, with sufficient accuracy one may write:

$$\overline{\chi_{2\tau}^2} = D_\text{y}\tau. \tag{9}$$

Formula (9) shows that the spectral line of a quartz oscillator has the form of a resonance curve; moreover, because we use dimensionless time, D_y is equal to the ratio of the line half-width to the fundamental frequency, i.e., it is equal to the relative frequency instability of the oscillator. Consequently:

$$\frac{\Delta\omega}{\omega_0} \approx D_\text{y} \approx 2\cdot 10^{-16}.$$

Fig. 2

If amplitude fluctuations did not affect the phase, then, under the direct action of the fluctuation forces, it would diffuse with diffusion coefficient $D_\text{n} \approx 10^{-14}$.

Then the spectral line would have a width of the order of the width of the line of a simple oscillator, calculated and measured by Bernstein ([4]).

We point out that measurements by the method ([4]), owing to the presence of technical instabilities, can be carried out only with small delay times, i.e., they give only D_{n}, which, as shown, does not determine the spectral line of a quartz oscillator.

An oscillator circuit with quartz in the anode and a locking circuit were considered analogously. The spectral-line width in both of these circuits proved to be of the same order as in the one considered above. In the locking circuit, in contrast to oscillator circuits, owing to the smallness of the direct influence of the shot effect on the oscillations of the quartz, the mean square of the phase run-up of the quartz over small time intervals grows more slowly than over large ones.

Comparison with experiment shows that shot and thermal fluctuations are not determining factors for the frequency stability of quartz oscillators, and that the fluctuation limit of stability of these oscillators has not yet been reached.

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Note: Figure translations are in progress. See original paper for figures.