G. D. Malyuzhinets

1958

SovietRxiv

Abstract

Full Text

MATHEMATICAL PHYSICS

G. D. Malyuzhinets

EXCITATION, REFLECTION, AND RADIATION OF SURFACE WAVES ON A WEDGE WITH GIVEN IMPEDANCES OF THE FACES

(Presented by Academician V. A. Fock on 10 IV 1958)

Suppose that a two-dimensional wave field $p(r,\varphi)$ $(p \sim e^{-i\omega t})$ in the wedge-shaped region $r > 0$, $-\Phi < \varphi < \Phi$ (Fig. 1) is described by the equation $\Delta p + k^2 p = 0$ and on the faces satisfies homogeneous boundary conditions of the third kind

$$(\partial p/r\, d\varphi)_\pm \pm ik \sin\vartheta_\pm\, p = 0 \quad (\varphi = \pm\Phi), \tag{1}$$

where $\sin\vartheta_\pm = z_0/z_\pm$; $z_0 = \rho c$ is the wave resistance of the medium; $z_\pm$ are the normal impedances of the boundaries $\varphi = \pm\Phi$. The constant quantities $\vartheta_\pm$ are the Brewster grazing angles, for which the reflection coefficient of a plane wave approaching an infinite plane with impedance $z_\pm$ at such a grazing angle vanishes. In the case of absorbing boundaries $0 < \operatorname{Re}\vartheta_\pm \le \pi/2$. The exact solution of the problem of diffraction of the plane wave

$$p_0 = \exp[-ikr\cos(\varphi - \varphi_0)] \tag{2}$$

by an absorbing wedge was obtained earlier ([1]) also for the case of oblique incidence on the edge, when

$$p_0 = \exp\{-ik[z\cos\theta + r\sin\theta\cos(\varphi - \varphi_0)]\}.$$

In the present article, after a shortened derivation of the solution (9), the special case of incidence of a wave at the Brewster angle is considered, which, with the orientation chosen here (Fig. 1), corresponds to $\varphi_0 = \Phi - \vartheta_+$ or $\varphi_0 = -\Phi + \vartheta_-$. If $\operatorname{Re}\vartheta_\pm = 0$, $\operatorname{Im}\vartheta_\pm < 0$, this corresponds to the arrival from infinity of nonattenuating surface waves. For $\operatorname{Im}\vartheta_\pm > 0$ it is not meaningful to call such waves surface waves, since

Fig. 1

Fig. 2

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

their amplitude in this case increases, rather than decreases, with distance from the surface.

In the general case, the solution of the problem of diffraction of the plane wave (2), satisfying the principle of absorption ([1,2]) and continuous at the boundary, is obtained in the form of a Sommerfeld integral (Fig. 2)

$$p(r, \varphi) = \frac{1}{2\pi i} \int_\gamma e^{-ikr\cos\alpha}\, s(\alpha + \varphi)\, d\alpha, \tag{3}$$

where the function

$$s(\alpha) - (\alpha - \varphi_0)^{-1} \tag{4}$$

is regular in the strip $|\operatorname{Re}\alpha| < \Phi + \varepsilon$, as follows. With the aid of (3) the boundary conditions (1) can be written in the form

$$-\frac{1}{2\pi i} \int_\gamma e^{-ikr\cos\alpha} (\sin\alpha \mp \sin\vartheta_\pm)\, s(\alpha \mp \Phi)\, d\alpha \equiv 0 \quad (r > 0).$$

For this identity to hold in the present case it is necessary and sufficient ([1,3,4]) that the coefficient of the exponential under the integral sign be an even function of α. Hence there arise two functional equations:

$$(\sin\alpha \mp \sin\vartheta_\pm) s(\alpha \mp \Phi) - (-\sin\alpha \mp \sin\vartheta_\pm) s(-\alpha \mp \Phi) = 0, \tag{5}$$

from which the function $s(\alpha)$, satisfying the regularity requirement (4), is to be determined.

By the substitution

$$s(\alpha) = \sigma(\alpha)\Psi(\alpha)/\Psi(\varphi_0) \tag{6}$$

equations (5) reduce to the simple equations

$$\sigma(\alpha \mp \Phi) - \sigma(-\alpha \mp \Phi) = 0,$$

whose solution satisfying the regularity condition (4) has the form

$$\sigma(\alpha) = \frac{\pi}{2\Phi} \cos\frac{\pi\varphi_0}{2\Phi} \Big/ \left(\sin\frac{\pi\alpha}{2\Phi} - \sin\frac{\pi\varphi_0}{2\Phi}\right). \qquad (7)$$

The auxiliary function $\Psi(\alpha)$ is a solution of equations (5) having no poles or zeros in the strip $|\operatorname{Re}\alpha| < \Phi + \varepsilon$. The logarithmic derivative of this function is readily found with the aid of (5) and Fourier integrals. Without giving the calculations, we give only the easily verified final expression (8) for the function $\Psi(\alpha)$ in the form of a product of special functions $\Psi_\Phi(\alpha)$, whose properties are listed below ([1]):

$$\Psi_\Phi(\alpha) = \prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\left[1 - \left(\frac{\alpha}{2\Phi(2n-1) + (\pi/2)(2m-1)}\right)^2\right]^{(-1)^{m+1}}$$
$$= \exp\left[\frac{i}{8\Phi}\int_0^{\alpha}\int_{-i\infty}^{i\infty} \tan\frac{\pi\nu}{4\Phi}\,\frac{d\nu\, d\mu}{\cos(\nu-\mu)}\right];$$

$$\frac{\Psi_\Phi(\alpha+2\Phi)}{\Psi_\Phi(\alpha-2\Phi)} = \operatorname{ctg}\frac{1}{2}\left(\alpha + \frac{\pi}{2}\right); \qquad \Psi_\Phi\left(\alpha+\frac{\pi}{2}\right)\Psi_\Phi\left(\alpha-\frac{\pi}{2}\right) = \Psi_\Phi^2\left(\frac{\pi}{2}\right)\cos\frac{\pi\alpha}{4\Phi};$$

$$\Psi_\Phi(\alpha+\Phi)\Psi_\Phi(\alpha-\Phi) = [\Psi_\Phi(\Phi)]^2\Psi_{\Phi/2}(\alpha).$$

For $|\operatorname{Im}\alpha| \to \infty$,

$$\Psi_\Phi(\alpha) = O\left[\exp\left|\frac{\pi \operatorname{Im}\alpha}{8\Phi}\right|\right].$$

For positive values of Φ, the zeros closest to the point $\alpha = 0$ and, respectively, the poles of the function are the points

$$\alpha = \mp\left(\frac{1}{2}\pi + 2\Phi\right) \quad \text{and} \quad \alpha = \mp\left(\frac{3}{2}\pi + 2\Phi\right).$$

For the rational ratio $4\Phi/\pi = n/m$ and

$$a(k,l) = \frac{\pi}{2}\left(\frac{2l-1}{n} - \frac{2k-1}{m}\right),$$

we have, respectively, for odd and even n:

$$\Psi_{\pi n/4m}(\alpha) = \prod_{k=1}^{m}\prod_{l=1}^{n}\left(\frac{\cos\frac{1}{2}a(k,l)}{\cos\frac{1}{2}\left|\alpha/n + a(k,l)\right|}\right)^{(-1)^l};$$

$$\Psi_{\pi n/4m}(\alpha) = \prod_{k=1}^{m}\prod_{l=1}^{n} \exp\left[\frac{(-1)^l}{\pi}\int_{a(k,l)}^{a(k,l)+\alpha/n} u \operatorname{ctg} u\, du\right].$$

Examples.

$$\Psi_{\pi/4}(\alpha) = \cos\frac{\alpha}{2}; \qquad \Psi_{\pi/2}(\alpha) = \exp\left[\frac{1}{4\pi}\int_0^\alpha \frac{2u - \pi\sin u}{\cos u}\, du\right];$$

$$\Psi_{3\pi/4}(\alpha) = \frac{\cos^{1/6}(\alpha-\pi)\cos^{1/6}(\alpha+\pi)}{\cos^2(\pi/6)\cos^{1/6}\alpha};$$

$$\Psi_{\pi}(\alpha) = \exp\left\{\frac{1}{8\pi}\int_0^\alpha \frac{i\pi\sin u - 4\pi\cos^{1/4}\pi\,\sin^{1/2}u - 2u}{\cos u}\, du\right\}.$$

With the aid of the function $\Psi_\Phi(\alpha)$, as is readily verified by writing the functional relations given above, the desired function $\Psi(\alpha)$, satisfying all the requirements mentioned earlier, is expressed by the formula

$$\begin{aligned}\Psi(\alpha) = \Psi_\Phi(\alpha + \Phi + \tfrac{1}{2}\pi - \vartheta_+)\Psi_\Phi(\alpha - \Phi - \tfrac{1}{2}\pi + \vartheta_-)\times\\ \times\Psi_\Phi(\alpha + \Phi - \tfrac{1}{2}\pi + \vartheta_+)\Psi_\Phi(\alpha - \Phi + \tfrac{1}{2}\pi - \vartheta_-),\end{aligned} \tag{8}$$

or, equivalently, by the formula*

$$\begin{aligned}\Psi(\alpha) = \left[\Psi_\Phi\left(\frac{\pi}{2}\right)\right]^4\left[\cos\frac{\pi}{4\Phi}(\alpha+\Phi-\vartheta_+)\cos\frac{\pi}{4\Phi}(\alpha-\Phi+\vartheta_-)\right]\times\\ \times\left\{\frac{\Psi_\Phi(\alpha + \Phi - \frac{1}{2}\pi + \vartheta_+)\Psi_\Phi(\alpha - \Phi + \frac{1}{2}\pi - \vartheta_-)}{\Psi_\Phi(\alpha + \Phi - \frac{1}{2}\pi - \vartheta_+)\Psi_\Phi(\alpha - \Phi + \frac{1}{2}\pi + \vartheta_-)}\right\}.\end{aligned}$$

The function $\Psi(\alpha)$ enters under the integral

$$p(r,\varphi) = \frac{1}{4\Phi i}\int_\gamma e^{-ikr\cos\alpha}\frac{\Psi(\alpha+\varphi)}{\Psi(\varphi_0)}\cos\frac{\pi\varphi_0}{2\Phi}\left(\sin\frac{\pi(\alpha+\varphi)}{2\Phi} - \sin\frac{\pi\varphi_0}{2\Phi}\right)^{-1} d\alpha, \tag{9}$$

which is the exact solution of the diffraction problem obtained by substituting (6) and (7) into (3). If, assuming $kr > 0$, the contour of integration γ is deformed into the two paths of steepest descent $\operatorname{Re}\alpha = \pm\pi - \operatorname{gd}(\operatorname{Im}\alpha)$** passing through the saddle points $\alpha = \pm\pi$ and shown by the dotted line in Fig. 2, then (9) becomes***

$$p(r,\varphi) = \frac{i}{4\Phi i}\int_{\gamma_1} e^{-ikr\cos\alpha}\frac{\Psi(\alpha+\varphi)\cos\frac{\pi\varphi_0}{2\Phi}\,d\alpha}{\Psi(\varphi_0)\left[\sin\frac{\pi(\alpha+\varphi)}{2\Phi}-\sin\frac{\pi\varphi_0}{2\Phi}\right]} + r_+ + r_- + \sum r_n, \quad (10)$$

where r are the residues at the poles of the integrand situated between the two lines of the contour γ_1, i.e. in the strip $-\pi - \operatorname{gd}(\operatorname{Im}\alpha) < \operatorname{Re}\alpha < \pi - \operatorname{gd}(\operatorname{Im}\alpha)$. It is assumed here that none of the poles lies on the contour γ_1. Since the possible values of the quantities φ and $\operatorname{Re}\varphi_0$ lie within $(-\Phi, \Phi)$, only the poles $\alpha_n = -\varphi + (-)^n\varphi_0 + 2n\Phi$, caused by the difference of the sines in the denominator, and the two poles $\alpha_\pm = -\varphi + (\pi + \Phi + \vartheta_\pm)$, belonging to the first and second factors of the function $\Psi(\alpha)$ (8), can fall into the indicated strip. By isolating one or another factor, the latter can be rewritten in one of the forms $\Psi(\alpha) = \Psi_\Phi(\alpha \pm \Phi \pm \frac{1}{2}\pi \mp \vartheta_\pm)\Psi^{(\pm)}(\alpha)$, where $\Psi^{(\pm)}(\alpha)$ denotes the product of the three remaining factors of $\Psi(\alpha)$. The residues $r_\pm$ at the poles $\alpha_\pm$ are readily determined with the aid of the relation

$$\Psi_\Phi\left[z \pm \left(2\Phi + \frac{3\pi}{2}\right)\right] = \mp\sin\frac{\pi(\pi+z)}{4\Phi}\,\operatorname{cosec}\frac{\pi z}{4\Phi}\,\Psi_\Phi\left(2\Phi - \frac{\pi}{2} \pm z\right),$$

* In paper (1), by $\Psi(\alpha)$ only the factor in braces is denoted, which remains regular as $|\operatorname{Im}\alpha| \to \infty$. In formula (71) of paper (3) a misprint was made. The product of cosines in square brackets is there erroneously replaced by

$$\sin[(\alpha - \Phi - \vartheta_+)\pi/4\Phi]\,\sin[(\alpha + \Phi + \vartheta_-)\pi/4\Phi].$$

** Gudermannian $\operatorname{gd} x = \arccos(1/\operatorname{ch} x)$.

*** The integral term here satisfies the radiation principle, remaining bounded for $\operatorname{Im} k > 0$. This requirement ensures uniqueness of the solution.

extracting the pole into a trigonometric factor. Taking into account whether or not the indicated poles fall into the given strip, for the corresponding additional terms in the right-hand side of (10) we obtain

$$r'_n = (-)^n \left\{ \frac{\Psi[(-)^n \varphi_0 + 2n\Phi]}{\Psi(\varphi_0)} \right\} \exp\{-ikr\cos[\varphi - (-)^n\varphi_0 - 2n\Phi]\}$$
$$\text{for } |\varphi - 2n\Phi - (-)^n\{\mathrm{gd}(\operatorname{Im}\varphi_0) + \operatorname{Re}\varphi_0\}| < \pi, \quad |\varphi| \le \Phi;$$
$$r_n = 0 \qquad \text{for } |\varphi - 2n\Phi - (-)^n\{\mathrm{gd}(\operatorname{Im}\varphi_0) + \operatorname{Re}\varphi_0\}| > \pi$$
$$(n = \dots, -2, -1, 0, 1, 2, \dots);$$
$$r_\pm = C_\pm \exp\{ikr\cos(\Phi + \vartheta_\pm \mp \varphi)\} \quad \text{for } 0 \le \Phi \mp \varphi < -\mathrm{gd}(\operatorname{Im}\vartheta_\pm) - \operatorname{Re}\vartheta_\pm;$$
$$r_\pm = 0 \quad \text{for } \Phi \mp \varphi > -\mathrm{gd}(\operatorname{Im}\vartheta_\pm) - \operatorname{Re}\vartheta_\pm;$$
$$C_\pm = 2\sin\frac{\pi^2}{4\Phi}\,\cos\frac{\pi\varphi_0}{2\Phi}\,\Psi_\Phi\left(2\Phi - \frac{\pi}{2}\right)\,\Psi^{(\pm)}[\mp(\Phi + \pi + \vartheta_\pm)]$$
$$\times\,\Psi^{-1}(\varphi_0)\left[\cos\frac{\pi(\pi + \vartheta_\pm)}{2\Phi} \mp \sin\frac{\pi\varphi_0}{2\Phi}\right]^{-1}.$$

Representing, for large distances $kr \gg 1$, the integral in (10) by the first term of the asymptotic expansion by means of the saddle-point method, we have

$$p(r,\varphi) \simeq \frac{\pi\cos\dfrac{\pi\varphi_0}{2\Phi}}{2\Phi\Psi(\varphi_0)}\left[\frac{\Psi(\varphi-\pi)}{\sin\dfrac{\pi(\varphi-\pi)}{2\Phi} - \sin\dfrac{\pi\varphi_0}{2\Phi}} - \frac{\Psi(\varphi+\pi)}{\sin\dfrac{\pi(\varphi+\pi)}{2\Phi} - \sin\dfrac{\pi\varphi_0}{2\Phi}}\right]\frac{e^{i(kr+\pi/4)}}{\sqrt{2\pi kr}} + r_+ + r_- + \sum r_n.$$

The first term is the cylindrical wave scattered by the edge of the wedge. The quantities r_n represent the incident ($n = 0$) and reflected waves. The quantities $r_\pm$ are surface waves, in general decaying ones, excited by the incident wave and going along the faces to infinity. Under the condition $\operatorname{Re}\vartheta_\pm = 0$, $\operatorname{Im}\vartheta_\pm < 0$, which in the acoustic case corresponds to a purely elastic impedance, these waves become nondecaying.

If a surface wave is incident from infinity, which corresponds to the particular case $\varphi_0 = \Phi - \vartheta_+$ or $\varphi_0 = -\Phi + \vartheta_-$ ($\operatorname{Im}\vartheta_\pm < 0$), then the quantities $C_\pm$ are the amplitude coefficients, respectively, of reflection of the surface wave and of its transmission to the other face. For integral values of the parameter $m = \pi/4\Phi$, no outgoing surface waves are formed, except in those cases when such a wave is at the same time the exact reflection from the other face of the incident wave. We note in passing that, as follows from the representation of $\Psi_{\pi/4m}(\alpha)$ written above and from (10), for such angles of opening the solution $p(r,\varphi) = \sum r_n$, corresponding to the geometrical approximation, is exact, since the integral in (10) vanishes owing to the periodicity of the integrand.

It should be noted that the excitation of a surface wave by an incident plane wave on the half-plane $\varphi = \Phi$ with prescribed value ϑ_+ was first treated exactly by another method by V. A. Fock for the case $\Phi = \frac{1}{2}\pi$, $\vartheta_- = 0$.

Acoustics Institute
Academy of Sciences of the USSR

Received
31 III 1958

REFERENCES

1. G. D. Malyuzhinets, *Some generalizations of the method of reflections in the theory of diffraction of sinusoidal waves*, Author' s abstract of doctoral dissertation, FIAN, 1950.

2. G. D. Malyuzhinets, DAN, **78**, No. 3, 439 (1951).

3. G. D. Malyuzhinets, *Acoustical Journal*, **1**, Nos. 2 and 3 (1955); errata, **2**, No. 2 (1956).

4. G. D. Malyuzhinets, DAN, **118**, No. 6 (1958).

5. V. A. Fock, *Matematicheskii sbornik*, **14** (56), Nos. 1-2 (1944).

Note: Figure translations are in progress. See original paper for figures.