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Abstract

Full Text

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ON BOUNDEDNESS AND STABILITY IN THE LARGE OF SOLUTIONS OF SOME NONLINEAR DIFFERENTIAL EQUATIONS

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Systems of nonlinear differential equations of "indirect automatic control" ([1])* are considered

$$\frac{dx}{d\sigma} = Ax + a\varphi(\sigma), \qquad \frac{d\sigma}{dt} = (b, x) - \rho\varphi(\sigma), \qquad \rho > 0. \tag{1}$$

The continuous function $\varphi(\sigma)$ satisfies the conditions

$$\varphi(0) = 0; \qquad \sigma\varphi(\sigma) > 0 \quad \text{for } \sigma \neq 0. \tag{2}$$

System (1) is called **stable in the large** if the trivial solution $x = 0,\ \sigma = 0$ is Lyapunov stable "in the small" and if, moreover, every solution satisfies $x \to 0,\ \sigma \to 0$ as $t \to \infty$.

Denote by $\lambda_j,\ j = 1, \dots, n$, the eigenvalues of the matrix A.

1°. In the preceding note ([2]) the author presented, in invariant form, A. I. Lur' e' s method ([1]) for determining conditions of stability in the large of system (1). (For the necessary additional proofs see also ([3]).) It was assumed there that all $\operatorname{Re}\lambda_i < 0$.

One of the necessary conditions for stability in the large for any function $\varphi(\sigma)$ satisfying conditions (2) has the form

$$\Gamma^2 \equiv \rho + (b, A^{-1}a) > 0. \tag{3}$$

We shall first clarify what is implied by the fulfillment of condition (3).

Lemma. *Suppose that a fixed solution x, σ exists on the interval $(0, \tau)$ and, if $\tau \neq \infty$, that it cannot be continued to values $t > \tau$. Let in system (1)* $\det A \neq 0$ *and let relation (3) hold. Then:*

I. From the boundedness of $x(t)$ as $t \to \tau$ there follows the boundedness of $\sigma(t)$ as $t \to \tau$, and also $\tau = \infty$.

II. If $x(t) \to 0$ as $t \to \tau$, then $\sigma(t) \to 0$ as $t \to \tau$, and the value itself is $\tau = \infty$.

Theorem 1. *Suppose that in system (1) the function* $\varphi(\sigma)$ *is bounded,* $|\varphi(\sigma)| \le \varphi_0$, $-\infty < \sigma < \infty$, *relation (3) holds, and* $\operatorname{Re}\lambda_j < 0$, $j = 1, \dots, n$. *Then every solution of system (1) exists on the infinite interval* $(0, \infty)$ *and is bounded as* $t \to \infty$. *Moreover, if* $\|e^{At}\| \le \alpha e^{-\beta t}$, $\alpha > 0$, $\beta > 0$, *then for* $t \ge 0$

$$\|x(t)\| \le \alpha e^{-\beta t}\|x(0)\| + \frac{\alpha\varphi_0}{\beta}\left[1 - e^{-\beta t}\right]\|a\|, \tag{4}$$

$$|\sigma(t)| \le |\sigma(0)| + 2\max_{t\ge 0}|(b, A^{-1}x)|. \tag{5}$$

Estimate (4) is obvious; only estimate (5) needs proof.

2°. Suppose that the matrix A has one zero eigenvalue and the remaining ones have negative real parts. Define uniquely the vectors x_0, y_0, z_0, a_1, b_1 from the equations $a = a_1 + y_0$, $Ay_0 = 0$, $(a_1, z_0) = 0$, $A^* z_0 = 0$, $Ax_0 = a_1$, $(x_0, z_0) = 0$, $b = b_1 + z_0$, $(b_1, y_0) = 0$.

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* We use here the same notation as in note ([2]). Capital Latin letters denote matrices, lowercase letters vectors, and Greek letters scalar quantities. Here the exceptions are: t is time; n is the order of vectors and matrices; i, j are indices; V is a Lyapunov function. All matrices, vectors, and numbers are assumed real.

According to the method of A. I. Lur' e, one must form the "resolving equations" ([2])

$$A^*U + UA = -uu^*, \qquad Ua + \rho u + \tfrac{1}{2}\rho b = 0, \tag{6}$$

where the matrix U is determined from the first equation (6) and substituted into the second equation (6), while the vector u is the unknown. In order that the first equation (6) have a solution, it is necessary that $(u, y_0) = 0$. In this case the matrix U is determined nonuniquely; however, this nonuniqueness does not affect the final form of the resolving equations. For stability in the large for any function $\varphi(\sigma)$ satisfying conditions (2), it is necessary that $\Gamma^2 \equiv \rho + (b, x_0) \ge 0$, $(y_0, z_0) < 0$. From equations (6) it is easy to derive the relation $(u, x_0) = \rho \pm \sqrt{\rho}\Gamma$, which, together with the relation $(u, y_0) = 0$, makes it possible to reduce the order of the resolving system of (scalar) equations by two units.

Theorem 2. *If the resolving equations (6) have a real solution* u *for the given vector* b *and for all vectors* b *sufficiently close to the given one, and the integrals*

$$\Phi_1 = \int_0^\infty \varphi(\sigma)\,d\sigma, \qquad \Phi_2 =$$

$$= \int_0^{-\infty} \varphi(\sigma)\,d\sigma,$$

diverge, then system (1) is stable in the large.

Let us note that, in the case when all $\operatorname{Re}\lambda_j < 0$, stability in the large holds independently of the divergence of the integrals Φ_1, Φ_2. Theorem 4 (see below) shows that now these conditions are essential.

Let $\mathfrak{R}$ be an n-dimensional space; $f(p,t)$ a dynamical system in it; θ the origin of coordinates; $f(\theta,t) \equiv \theta$. Suppose that in some domain $G \ni \theta$ (possibly $G = \mathfrak{R}$) there exists a Lyapunov function, i.e., a continuous function satisfying the conditions:

1) $V(p) > 0$ for $p \neq \theta$, $\quad V(\theta) = 0$;

2) $V[f(p,t)]$ is a nonincreasing function of t as long as $f(p,t) \in G$

and sometimes the additional condition:

3) for every $p \neq \theta$

$$\lim_{t\to\infty} V[f(p,t)] < V[f(p,0)] = V(p).$$

Denote by ξ_γ the connected component, containing the origin, of the open set $\xi\{V(p) < \gamma\} \subset G$. Suppose that for $\gamma < \gamma_0$ the sets ξ_γ are bounded. Put

$$G_0 = \bigcup_{\gamma<\gamma_0} \xi_\gamma$$

(the set G_0 may, of course, be unbounded).

Theorem 3. a) *If the function $V(p)$ satisfies conditions 1), 2), then the trivial motion $p \equiv \theta$ is Lyapunov stable. If $p \in G_0$, then the trajectory $f(p,t)$ is bounded as $t \to \infty$.*

b) *If the function $V(p)$ satisfies conditions 1), 2), 3), then the trivial motion $p \equiv \theta$ is asymptotically Lyapunov stable; if $p \in G_0$, then $f(p,t) \to \theta$ as $t \to \infty$. If, in addition, $G = \mathfrak{R}$, then for every p either $f(p,t) \to \theta$, or $\|f(p,t)\| \to \infty$ as $t \to \infty$* **.

By the substitution

$$x = S\begin{pmatrix} y \\ \eta \end{pmatrix}$$

with a suitably chosen matrix S, system (1) is reduced to the system ***

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* It can be shown that the verification of the conditions of the theorem can be carried out, unless certain exceptional cases occur, by a finite number of rational algebraic operations. The set of systems (1) for which exceptional cases occur has measure zero in the coefficient space. There are certainly no exceptional cases for $n \leq 4$.

** This theorem, which is a refinement of the corresponding Lyapunov theorem, is carried over verbatim (with the replacement of the word "boundedness" by "compactness") to the case when $\mathfrak{R}$ is a locally connected metric space, $\|p\| = \rho(p, \theta)$, and all spheres $\|p\| = \varepsilon$ are sufficiently small and compact. Instead of the equilibrium point θ, one could speak of an arbitrary invariant set $\mathfrak{M} = \{\theta\}$.

*** Here the vectors and matrices have order $n-1$.

$$\frac{dy}{dt} = Qy + q\varphi(\sigma), \qquad \frac{d\eta}{dt} = \varphi(\sigma), \qquad \frac{d\sigma}{dt} = (r, y) - \varkappa\eta - \rho\varphi(\sigma), \quad \varkappa > 0,\ \rho > 0, \tag{7}$$

where the eigenvalues of the matrix Q have negative real parts. The Lyapunov function

$$V = (Kx, x) + \int_0^\sigma \varphi\, d\sigma$$

in this basis will necessarily have the form

$$V = (Hy, y) + \psi\eta^2 + \int_0^\sigma \varphi\, d\sigma, \qquad H > 0,\ \psi > 0$$

(the product $h^* y\eta$ is absent). The derivative dV/dt can be reduced to the form

$$dV/dt = -(Cy, y) - \frac{1}{\rho}\,[\rho\varphi + (g, y)]^2.$$

Theorem 4. *Suppose that there exists a finite derivative* $\varphi'(\sigma)$, *at least one of the integrals Φ_1, Φ_2 converges, and there exists a Lyapunov function V of the indicated form, where the matrix $C > 0$**. Then, for system (7), every solution is continuable for $t \to \infty$, and also:*

a) the trivial solution is asymptotically stable in the sense of Lyapunov; the domain $V < \min[\Phi_1, \Phi_2]$ is entirely contained in the domain of asymptotic stability;

b) for every solution, as $t \to \infty$,

$$y \to 0, \quad \eta \to \eta_\infty, \quad \sigma = -\varkappa \int_0^t \eta(t)\, dt + \sigma_* + o(1). \tag{8}$$

*If $|\varphi(\sigma)| \leq \varphi_0$ and $\Phi_1 \neq \infty$ ($\Phi_2 \neq \infty$), then for any $y(0), \sigma(0)$ there is an $\eta(0) < 0$ such that $\eta_\infty < 0$ (respectively, an $\eta(0) > 0$ such that $\eta_\infty > 0$)***.

3°. If the matrix A has two or more zero eigenvalues, then already the linearized system (1) with $\varphi(\sigma) = \mu\sigma$ is unstable for sufficiently small $\mu > 0$. Therefore the problem of stability in the large of system (1), for any function $\varphi(\sigma)$ satisfying conditions (2), is in this case solved negatively.

4°. The systems of M. A. Aizerman (4), as well as more general systems of "direct automatic regulation" (1) with one nonlinearity $\psi(\sigma)$, when the linearized system with $\psi(\sigma) = \mu\sigma$ has a stability zone of the form (μ_0, ∞), when on the boundary of the stability zone $\mu = \mu_0$ there is at least one zero root, and the function $\psi(\sigma)$, as usual, satisfies the conditions: $\psi(0) = 0$; for $\sigma \neq 0$, $\sigma\psi(\sigma) > \mu_0\sigma^2$, is reduced to systems (1) with the function $\varphi(\sigma) = \psi(\sigma) - \mu_0\sigma$.

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[1] A. I. Lur' e, *Some Nonlinear Problems of the Theory of Automatic Regulation*, 1951. [2] V. A. Yakubovich, DAN, **117**, No. 1 (1957). [3] V. A. Yakubovich, Vestn. LGU, **19**, issue 4 (1957). [4] M. A. Aizerman, *Lectures on the Theory of Automatic Regulation*, 1956, p. 220. [5] V. A. Pliss, DAN, **120**, No. 4 (1958).

* This requirement may be replaced by the requirement of uniqueness and continuous dependence of solutions on the initial data.

** If the resolving equations (6) have a solution of the required form (see Theorem 2), then such a Lyapunov function exists.

*** V. A. Pliss previously showed, by an example for $n = 2$, the absence of stability in the large in the presence of a Lyapunov function with a convergent integral Φ_1 or Φ_2. For this example, the system of resolving equations (6) reduces to one quadratic equation. Theorem 4 gives an additional domain of asymptotic stability and relation (8), and also shows that an analogous situation occurs for any system (1) in the presence of one zero root. The author takes this opportunity to express gratitude to V. A. Pliss, who provided him with this interesting example.

Note: Figure translations are in progress. See original paper for figures.