ON ONE GENERAL METHOD OF OBTAINING SUBGROUPS AND FACTORIZATIONS OF FINITE GROUPS

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Abstract

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S. A. CHUNIKHIN

ON ONE GENERAL METHOD OF OBTAINING SUBGROUPS AND FACTORIZATIONS OF FINITE GROUPS

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§ 1. In the present paper we give some general criteria, obtained by us, for the existence of subgroups and factorizations in finite groups, based on the use of divisors of the indices of chief series. The main results of the paper are Theorems 1 and 4.

Theorem 1 is a sufficient criterion for the existence of subgroups; as its particular cases there follow the following theorems on the existence of subgroups: those of Sylow, P. Hall (for soluble groups), Schur (on factorizations of groups), S. A. Chunikhin (Theorem 14 of paper (^{1}) for the product of a relatively prime and a separable divisor of the order of a group), S. A. Chunikhin (Theorem 1 of paper (^{2}) for Π-relatively prime divisors of the order of a group), Theorems E2.2 and E2.3, and also, in part, Theorem E1* of paper (^{3}) of P. Hall. With the exception of the theorem on Π-relatively prime divisors of the order of a group, all these theorems can also be obtained from Theorem 2 of the present paper.

In Theorem 4 we establish the existence in finite groups of a new type of factorization with pairwise permutable factors. Particular cases of factorizations of this kind will be the factorizations of our papers (^{1,2}). In addition, Theorem 4 also implies the factorization assertion of Theorem A3 of P. Hall (^{3}).

§ 2. Let $\mathfrak{G}$ be an arbitrary finite group of order $(\mathfrak{G}) = g$, and let $\mathfrak{E}$ be its identity subgroup. If $g > 1$, then let

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \cdots \supset \mathfrak{G}_\lambda = \mathfrak{E}$$

be some chief series of $\mathfrak{G}$ with sequence of indices $h_1, h_2, \dots, h_\lambda$. If $g = 1$, then as a chief series of the group $\mathfrak{E}$ we shall regard the series $\mathfrak{E}, \mathfrak{E}$ with sequence of indices $h_1 = 1$.

We shall call an **admissible divisor** of some index of a chief series of the group $\mathfrak{G}$ a divisor which is either the highest divisor of the index that is a power of some prime number, or is equal to unity, or coincides with this index itself.

Let from the sequence $1, 2, \dots, \lambda$ there be chosen in some way a nonempty subsequence ω of the form $i_1, i_2, \dots, i_k$. Let, further, there be given some function f assigning to each $i_j \in \omega$ some admissible divisor f_{i_j} of the index h_{i_j}. Let also

$$\delta_{\omega,f} = f_{i_1} f_{i_2} \cdots f_{i_k}.$$

It is obvious that $\delta_{\omega,f} = h$ will be a divisor of g.

Definition 1. Every number h of the form $\delta_{\omega,f}$ will be called an **indexial** of the group $\mathfrak{G}$.

If n is some natural number, then by $\Pi(n)$ we shall denote the set of all prime divisors of n.

Theorem 1. *If h is some indexial of an arbitrary finite group $\mathfrak{G}$, then $\mathfrak{G}$ has at least one subgroup of order hc, where $\Pi(c)$ is contained in $\Pi(h)$.*

Let now w have the form $w = \{\beta, \beta + 1, \dots, \lambda\}$. We construct the function f as follows: we set f_β arbitrarily equal to one of the admissible divisors h_β; for $i > \beta$ we set f_i equal to one of those admissible divisors h_i which is divisible by $(f_\beta f_{\beta+1} \cdots f_{i-1}, h_i)$—let us note that such a divisor always exists, since, for example, the index h_i itself may be taken as it.

It is obvious that, for given w and f_β, several such functions can be constructed. Let f be one of them. Put $\vec{\delta}_{w,f} = f_\beta f_{\beta+1} \cdots f_\lambda$. It is obvious that if h is a number of the form $\vec{\delta}_{w,f}$ for a given chief series of the group, then h need not be a number of this type for another chief series of the same group.

Definition 2. A number h of the form $\vec{\delta}_{w,f}$ will be called a **directed index** of the group $\mathfrak{G}$.

Theorem 2. *If h is a directed index of an arbitrary finite group $\mathfrak{G}$, then $\mathfrak{G}$ has at least one subgroup of order h.*

§ 3. Let us consider several special cases of Theorem 1.

1) Let h be a full-block divisor (^{1}) and s a separable divisor (^{1}) of the order g of the group $\mathfrak{G}$, and let $(h, s) = 1$. From the definition of these divisors (^{1}) and the properties of the factors of chief series it follows that the greatest common divisor of any index of a chief series of the group and of the number hs will be equal either to 1, or to the highest divisor of the index which is a power of some prime number, or to this index itself. Hence the product of all these greatest common divisors will be an index of the group. On the other hand, it is not hard to verify that this product coincides with hs. Hence, by Theorem 1, we conclude that $\mathfrak{G}$ has a subgroup of order hsc, where $\Pi(c) \subseteq \Pi(hs)$. But from the definition of the numbers h and s it also follows that

$$\left(hs, \frac{g}{hs}\right) = 1.$$

Therefore $c = 1$.

Consequently, Theorem 14 of the paper (1) is a special case of Theorem 1. But in § 11 of the paper (1) we showed that special cases of the indicated Theorem 14 are the theorems on the existence of subgroups of Sylow, Ph. Hall (for solvable groups), S. A. Chunikhin (for Π-separable groups and full-block divisors of the order of a group), and Schur' s factorization theorem. Thus, all the theorems listed are special cases of Theorem 1.

2) Let h be a Π-full-block divisor (2) of the order g of the group $\mathfrak{G}$, and let h' be the greatest Π-divisor (2) of h. Then (see (2)) we have

$$\left(h', \frac{g}{h'}\right) = 1.$$

From the definition of a Π-full-block divisor and the properties of the factors of a chief series it follows that h will be the product of certain indices of a chief series of $\mathfrak{G}$, i.e. that h will be an index of $\mathfrak{G}$. By Theorem 1, $\mathfrak{G}$ has a subgroup of order hc, where $\Pi(c) \subseteq \Pi(h)$. Since hc divides g, $(h', \frac{g}{h'}) = 1$, and $\Pi(c) \subseteq \Pi(h)$, it follows that c will be a Π-number. We have obtained Theorem 1 of the paper (2).

3) Let us consider the case of Theorem E2.2 of Ph. Hall (3). It is not difficult to verify that the conditions of Theorem E2.2 are satisfied also by the indices of a chief series of the group. Consider those of them which are ω-composite (for the definition of an ω-composite number see (2)). The greatest ω-divisors of such indices will, in the case of Theorem E2.2, either be equal to these indices or be powers of prime numbers from ω. Therefore the product h of all these greatest ω-divisors will be an index of $\mathfrak{G}$. Since all ω-composite indices of the chief series of $\mathfrak{G}$ were considered, we have

$$\left(h, \frac{g}{h}\right) = 1.$$

By Theorem 1, $\mathfrak{G}$ has a subgroup of order hc, $\Pi(c) \subseteq \Pi(h)$. Since

$$\left(h, \frac{g}{h}\right) = 1,$$

we have $c = 1$. Thus, Theorem E2.2 is a special case of Theorem 1.

In a similar way one can obtain from Theorem 1 also Theorem E2.3 and the assertion on the existence of the subgroup $\mathfrak{L}$ of Theorem E1* of Ph. Hall (3).

§ 4. **Theorem 3.** Let a group $\mathfrak{G}$ of order g have a normal divisor $\mathfrak{G}_1$ of order g_1, and let p^α, $\alpha > 0$, and q^β, $\beta > 0$, be the highest powers of distinct or identical primes p and q dividing respectively $\dfrac{g}{g_1}$ and g_1. Then $\mathfrak{G}$ has at least one subgroup of order $p^\alpha q^\beta$.

§ 5. Denote the sequence $h_1, h_2, \dots, h_\lambda$ by $\mathfrak{S}_\mathfrak{G}$. Indices occupying **different** positions in this sequence will be regarded as **different** (even if they are arithmetically equal).

Definition 3. Let M be some subsequence of the sequence $\mathfrak{S}_\mathfrak{G}$. Introduce (cf. with (2)) the symbol $|M|$, setting it, when M is nonempty, equal to the product of all elements of M, and when M is empty, to the number 1. In particular, if M contains only one element m, then $|M| = m$.

Theorem 4. Let $\mathfrak{G}$ be an arbitrary finite group. Then to every collection of such subsets $M_1, M_2, \dots, M_\mu$, $\mu \geqslant 1$, of the set $\mathfrak{S}_\mathfrak{G}$, whose union is $\mathfrak{S}_\mathfrak{G}$ and for which $|M_1| = m_1$, $|M_2| = m_2, \dots,$ $|M_\mu| = m_\mu$, there corresponds a representation of the group $\mathfrak{G}$ in the form of a product

$$\mathfrak{G} = \mathfrak{M}_1 \mathfrak{M}_2 \cdots \mathfrak{M}_\mu,$$

where $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_\mu$ are, for $\mu > 1$, certain pairwise permutable subgroups of $\mathfrak{G}$, respectively of orders $m_1 c_1, m_2 c_2, \dots, m_\mu c_\mu$, where for each $i = 1, 2, \dots, \mu$ the set $\Pi(c_i)$ is contained in $\Pi(m_i)$.

Recalling the definition of composition (1) and Π-composition blocks (2), and using the properties of the factors of a chief series, from Theorem 4 we directly obtain factorizations of finite groups from works (1,2) (the same also applies to the factorization from Theorem A3 of P. Hall (3)).

Belorussian Institute
of Railway Transport Engineers

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