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Abstract

Full Text

MATHEMATICS

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ON THE SOLUTION OF EXTREMAL PROBLEMS FOR QUASICONFORMAL MAPPINGS BY THE METHOD OF VARIATIONS

(Presented by Academician M. A. Lavrent'ev, March 6, 1958)

Consider the following problem. In the class of q -quasiconformal mappings $w = w(z)$, one seeks the maximum of the real function $F(z_1, \dots, z_k, w_1, \dots, w_k)$, $w_n = w(z_n) = u_n + iv_n$.

We shall assume that the mapping is made onto some canonical domain with a normalization ensuring the uniqueness of the mapping function for the prescribed characteristics. Let, for example, $w = w(z)$ be a mapping of the disk $|z| \leq 1$ onto the disk $|w| \leq 1$, $w(0) = 0$, $w(1) = 1$. We shall assume the function F to be continuously differentiable with respect to u_n, v_n . By virtue of the compactness of the class of q -quasiconformal mappings ⁽¹⁾, there exists a general q -mapping $w = f(z)$ for which F assumes its maximum value. Let us first suppose, for simplicity, that the extremal mapping is sufficiently smooth, and try actually to find it. Let the characteristics of the inverse mapping be $p(w), \theta(w)$ (i.e., an infinitely small circle with center at the point $z = f^{-1}(w)$ passes into an ellipse with center at the point w , with ratio of the major semiaxes $p(w) \geq 1$ and angle $\theta(w)$ between the major semiaxis and the x -axis).

We shall consider the complex characteristic

$$h(w) = \frac{p-1}{p+1} e^{2i\theta}.$$

Since $p(w) \leq q$, we have

$$|h(w)| \leq \frac{q-1}{q+1} = h_0.$$

Let, as usual,

$$\frac{\partial}{\partial w} = \frac{1}{2} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{w}} = \frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right);$$

then

$$dz = z_w dw + z_{\bar{w}} d\bar{w}.$$

It is not hard to verify that

$$h = -z_{\bar{w}} : z_w,$$

i.e.

$$dz = z_w [dw - h d\bar{w}].$$

Subject the disk w to a variation according to the formula

$$\omega = \left\{ w \left[1 + (\omega'_* - 1) \left(\frac{h_1(\zeta)}{\zeta(1-\zeta)(w-\zeta)} + \frac{\bar{h}_1(\zeta)}{\bar{\zeta}(1-\bar{\zeta})(1-w\bar{\zeta})} \right) \frac{d\sigma_\zeta}{\pi} \right] \right\}. \quad (1)$$

This variation represents a mapping of the disk $|w| \leq 1$ onto the disk $|\omega| \leq 1$, conformal outside the circle with center at the point $w = \zeta$ and area $d\sigma_\zeta$ (2). Formula (1) is valid up to quantities of second order of smallness and is applicable to points lying outside the indicated circle. Inside the circle the variation has the constant characteristic

$$h_1(\zeta) = \frac{p_1 - 1}{p_1 + 1} e^{2i\theta_1} = o(1).$$

If r denotes the radius of the varied circle, then for $|w - \zeta| < r$ in formula (1) one must replace

$$\frac{1}{w - \zeta}$$

by

$$\frac{1}{r^2} (\bar{w} - \bar{\zeta}).$$

It is not hard to see that inside the varied circle $d\omega$ has the form

$$d\omega = (1 - \varepsilon_1) dw + (1 + \varepsilon_2) h_1 d\bar{w},$$

where ε_1 and ε_2 are small quantities of order no higher than $|h_1|$. Denote by \tilde{h} the characteristic of the varied mapping.

Then for $(w - \zeta) < r$

$$\begin{aligned} dz &= z_w (dw - h d\bar{w}) = z_w [(1 + \varepsilon_1) dw + (1 + \varepsilon_2) h_1 d\bar{w} - h(1 + \bar{\varepsilon}_1) d\bar{w} - h h_1 (1 + \bar{\varepsilon}_2) dw] \\ &= [z_w (1 - \varepsilon_1 - \bar{h} h_1 - \bar{h} h_1 \varepsilon_2) dw - (h + \bar{h} \bar{\varepsilon}_1 - h_1 - h_1 \varepsilon_2) d\bar{w}]. \end{aligned}$$

Therefore

$$|\tilde{h}| = \left| \frac{h + \bar{h} \bar{\varepsilon}_1 - h_1 - h_1 \varepsilon_2}{1 + \varepsilon_1 - h h_1 - \bar{h} h_1 \varepsilon_2} \right| \approx |h + \bar{h} \bar{\varepsilon}_1 - h \varepsilon_1 - h_1 + h^2 h_1|$$

$$\approx |h| \left[1 + \operatorname{Re} \left(h h_1 - \frac{h_1}{h} \right) \right],$$

or

$$|\tilde{h}| = |h| - |h_1|(1 - |h|^2) \cos 2(\theta - \theta_1). \quad (2)$$

Let us now compute the increment of the function F , taking into account that, since F is real, $F_{\bar{w}_n} = \overline{F_{w_n}}$. We have

$$dF = \sum_1^k (F_{w_n} dw_n + \overline{F_{w_n}} d\bar{w}_n),$$

where dw_n , by virtue of (1), has the form

$$dw_n = w_n(w_n - 1) \left\{ \frac{h_1(\zeta)}{\zeta(1 - \zeta)(w_n - \zeta)} + \frac{\bar{h}_1(\zeta)}{\bar{\zeta}(1 - \bar{\zeta})(1 - w_n\bar{\zeta})} \right\} \frac{d\sigma_\zeta}{\pi}. \quad (3)$$

Therefore dF can be represented in the form $2|A||h_1| \cos 2(\theta_1 - \varphi)$, where A is a rational function of ζ with poles at the points $\zeta = 0, 1, w_1, \dots, w_k, \frac{1}{\bar{w}_1}, \dots, \frac{1}{\bar{w}_k}$, and $\varphi = -\frac{1}{2} \arg A(\zeta)$ (we leave aside the case $A \equiv 0$, corresponding to the presence of a stationary value of F ($F_{w_n} = 0$, $n = 1, \dots, k$)). Comparing (2) and (3), we see that in the case $|h(\zeta)| < h_0$ a variation with arbitrary sufficiently small h_1 is admissible, and dF can have either sign. Therefore, in the case of an extremal mapping the characteristic $h(w)$ must be as follows:

$$|h(w)| = h_0 = \frac{q - 1}{q + 1}, \quad \theta = \frac{1}{2} \arg h'(w) = -\frac{1}{2} \arg A(w), \quad (4)$$

i.e. the mapping belongs to the class of extremal mappings considered by Teichmüller (3).

Let us return to the question of the existence of an extremal function. According to what was said at the beginning of the article, there exists an extremal function belonging to the closure of the class of continuously differentiable quasiconformal mappings. This function $z = z(w)$ will be differentiable almost everywhere and, since its derivatives are, obviously, measurable, for an arbitrary $\varepsilon > 0$ one can indicate a set D_ε , $\operatorname{mes} D_\varepsilon > \pi - \varepsilon$, on which the derivatives are continuous. Let $\zeta \in D_\varepsilon$ be a point of density of the set D_ε .

Consider a neighborhood K_δ of the point ζ so small that for all $w \in K_\delta \cap D_\varepsilon$, $|h(w) - h(\zeta)| < \delta$, and, moreover, so that $\operatorname{mes}(K_\delta \cap D_\varepsilon) > (1 - \delta) \operatorname{mes} K_\delta$.

Varying the characteristic in the disk K_δ , we shall apply the same reasoning as above; however, the characteristic on the set $K_\delta \cap D_\varepsilon$ may increase. Therefore the w -plane must be subjected to an additional mapping, conformal everywhere

except on the set $K_\delta \cap D_\varepsilon$, and on the indicated set transforming the ellipses present there with $p > q$ into some ellipses with $p \leq q$. This is possible by virtue of the existence theorem ⁽¹⁾. The effect of the additional variation is small in comparison with the principal variation, as follows from the lemma:

Lemma. Let the function $w = w(z)$, $w(0) = 0$, $w(1) = 1$, carry out a quasiconformal mapping of the disk $|z| \leq 1$ onto the disk $|w| \leq 1$; let $|h(z)| \leq h_0$ and $h(z) \neq 0$ only for $|z - \zeta| < r$, $\zeta \neq 0, 1$, and only on a set of area $\varepsilon\pi r^2 = \varepsilon d\sigma_\zeta$.

Then, for $|z - \zeta| > r$,

$$|\omega(z) - z| \leq \lambda(\varepsilon, h_0) f(\zeta, z). \quad (5)$$

where $\lim_{\varepsilon \rightarrow 0} \lambda(\varepsilon, h_0) = 0$.

The proof of the lemma follows easily from (4) and the general properties of conformal mappings.

Thus the set $K_\delta \cap CD_\varepsilon$ may be neglected, and the same arguments as in the case of a discontinuous characteristic $h(w)$ lead to relations (4) at all density points of D_ε , i.e., ultimately, almost everywhere in the disk $|w| < 1$. Since, by the characteristics defined almost everywhere, the general q -mapping is determined uniquely, it follows that the extremal function coincides with the mapping determined by condition (4). Thus the question of the existence and uniqueness of the extremal function is completely resolved. Using the indicated method, and also the fact that the extremal mapping for $1 \leq q \leq 1 + \varepsilon$, $\varepsilon = o(1)$, can be written in the form of an integral from (1), we obtain the following assertions:

Theorem 1. Let $w = w(z)$ map the disk $|z| \leq 1$ quasiconformally onto the disk $|w| \leq 1$, and let $w(0) = 0$, $w(1) = 1$, $1 \leq p(z) \leq 1 + \varepsilon$. Then

$$|w(z) - z| \leq \varepsilon \frac{8}{\pi} \int_0^1 K(r^2) dr \simeq 4.5 \varepsilon, \quad (6)$$

where $K(\varepsilon)$ is the complete elliptic integral of the first kind. The constant

$$M = \frac{8}{\pi} \int_0^1 K(r^2) dr$$

cannot be improved.

Theorem 2. Let $w = w(z)$ map the disk $|z| \leq 1$ quasiconformally onto the disk $|w| \leq 1$ with normalization $w(0) = 0$, or map the z -plane onto the w -plane with normalization $w(\infty) = \infty$. Then

$$\rho < \left| \frac{w(z_0 + re^{i\varphi_1}) - w(z_0)}{w(z_0 + re^{i\varphi_2}) - w(z_0)} \right| < \rho(1 + M\varepsilon), \quad (7)$$

where the constant

$$M = \frac{8}{\pi} \int_0^1 K(r^2) dr$$

coincides with the constant of Theorem 1 and cannot be improved. Moreover,

$$\left| \arg \frac{w(z_0 + re^{i\varphi_1}) - w(z_0)}{w(z_0 + re^{i\varphi_2}) - w(z_0)} - (\varphi_1 - \varphi_2) \right| < M\varepsilon, \quad (8)$$

where M is the same constant, which also cannot be improved.

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