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## Abstract

## Full Text

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## MATHEMATICS

I. V. GELMAN

# ON SOME FUNCTIONAL SPACES AND THEIR APPLICATIONS TO VARIATIONAL PROBLEMS

*(Presented by Academician I. V. Smirnov, 5 V 1958)*

In the present note the embedding theorems of S. L. Sobolev <sup>(1)</sup> are extended to the case of spaces of functions whose  $l$ -th generalized derivatives belong to a certain Orlicz space.

1. Let  $M(u)$  and  $N(v)$  be two mutually complementary  $N'$ -functions;  $L_M(\Omega)$  and  $L_M^*(\Omega)$  are the class and the Orlicz space generated by the function  $M(u)$ ;  $E_M(\Omega)$  is the closure of the set of bounded functions in the metric  $\| \cdot \|_M$  ( $\Omega$  is an  $n$ -dimensional bounded domain) <sup>(2)</sup>.

We shall say that a function  $u(x)$  ( $x \in \Omega$ ) belongs to the class  $F_M^{(l)}$  if all its generalized derivatives of order  $l$  (in the sense of S. L. Sobolev) exist and belong to the space  $L_M^*(\Omega)$ . Let  $S^{(l)}$  be the set of polynomials of degree not higher than  $(l-1)$ ; let  $\Phi_M^{(l)}$  be the collection of cosets of the set  $F_M^{(l)}$  modulo the set  $S^{(l)}$ . If  $\Pi$  is a projection operator ( $\Pi^2 u = \Pi u$ ) from  $F_M^{(l)}$  onto  $S^{(l)}$ , then

$$u = \Pi u + u^*, \quad \Pi u \in S^{(l)}, \quad u^* \in \Phi_M^{(l)} \quad (u \in F_M^{(l)}). \quad (1)$$

Put

$$D^{(l)}(u) = \sum_{i_1, \dots, i_n} |\partial^l u / \partial x_{i_1} \dots \partial x_{i_n}|$$

and let

$$\Pi u = \sum_{\sum \alpha_i \leq l-1} a_{\alpha_1, \dots, \alpha_n} x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

By the same symbol  $F_M^{(l)}$  we shall denote the space of functions of the class  $F_M^{(l)}$ , whose norms we define by the equality

$$\|u\|_{F_M^{(l)}} = \sum_{\sum \alpha_i \leq l-1} |a_{\alpha_1, \dots, \alpha_n}| + \|D^{(l)}u\|_M. \quad (2)$$

Let the domain  $\Omega$  be star-shaped with respect to a ball  $K$ ; let  $\omega^{(s)}$  be a manifold of  $s$  dimensions from  $\Omega$  ( $s \leq n$ ; for  $s = n$  we assume  $\omega^{(s)} = \Omega$ ), which by a coordinate transformation introducing only a finite distortion of distances can be transformed into a flat one. Let  $P(x_1, \dots, x_n) \in \Omega$ ,  $Q(y_1, \dots, y_s) \in \omega^{(s)}$ ,

$$\rho_1 = \left( \sum_{i=1}^s y_i^2 \right)^{1/2}, \quad \rho_2 = \left( \sum_{i=1}^n x_i^2 \right)^{1/2}, \quad \rho_3 = \left( \sum_{i=s+1}^n x_i^2 \right)^{1/2}.$$

The following embedding theorems belong to E. P. Kalugina <sup>(3)</sup>:

**Theorem 1.** If  $u(x) \in F_M^{(l)}$  and

$$\int N [\rho_2^{l-n-k}] dx_1 \dots dx_n < \infty,$$

then  $u(x)$  has continuous derivatives up to order  $k$  inclusive ( $0 \leq k < l$ ).

**Theorem 2.** Let  $0 \leq k < l$ ,  $s > n - l + k$  and  $0 < k_1 < n - l + k$ ,  $\delta > 0$ , such that

$$\int N [\rho_2^{l-n-k+k_1}] dx_1 \dots dx_n < \infty, \quad \int M_1 [\rho_1^{-k_1-\delta}] dy_1 \dots dy_s < \infty$$

( $M_1(u)$  is some  $N'$ -function). If  $u(x) \in F_M^{(l)}$ , then  $u(x) \in F_{M_1}^{(k)}(\omega^{(s)})$ .

**Theorem 2'.** Let  $0 \leq k < l$ ,  $s > n - l + k$ . If the function  $u(x) \in F_M^{(l)}$ , then all its derivatives of order  $k$  belong to the space  $L_M^*(\omega^{(s)})$ .

Carrying out estimates analogous to those of V. P. Il' in (4), we extend Theorem 2 to the case of manifolds of dimension less than  $n - l + k$ .

**Theorem 3.** Let  $0 \leq k < l$ ,  $\delta > 0$ ,  $k_1 > 0$ ,  $k_2 > 0$ ,  $s > k_1 + k_2$ , and let  $k_1 + k_2 \leq n - l + k$  (the equality sign is possible only when  $s = n$ ), so that

$$\int N [\rho_3^{l-n-k+k_1+k_2}] dx_{s+1} \dots dx_n < \infty, \quad \int N [\rho_1^{-k_2}] dy_1 \dots dy_s < \infty,$$

$$N(V) > |V|^{(n-s)(n-l+k-k_1-k_2)-\delta(n-s)/2k_2(n-l+k-k_1-k_2)}, \quad \int M_1 [\rho_1^{-k_1\delta}] dy_1 \dots dy_s < \infty.$$

If  $u(x) \in F_M^{(l)}$ , then  $u(x) \in F_{M_1}^{(k)}(\omega^{(s)})$ .

**Remark.** It is not hard to prove that under the hypotheses of Theorems 2 and 3 the  $k$ -th derivatives of the function  $u(x)$  belong to the space  $E_{M_1}(\omega^{(s)})$  ( $E_M(\omega^{(s)})$  under the hypotheses of Theorem 2').

**Theorem 4.** The embedding operator is bounded:

$$\left\| \partial^k u / \partial x_1^{k_1} \dots \partial x_n^{k_n} \right\| \leq a \|U\|_{F_M^{(l)}} \quad (3)$$

( $a > 0$  is a constant; in the left-hand side of inequality (3) the norm is taken in the space  $C$ ,  $L_{M_1}^*(\omega^{(s)})$ , or  $L_M^*(\omega^{(s)})$ ).

**Theorem 5.** If  $u_m(x) \in F_M^{(l)}$  ( $m = 1, 2, \dots$ ) and the higher derivatives

$$\{\partial^l u_m / \partial x_1^{l_1} \dots \partial x_n^{l_n}\} \quad (m = 1, 2, \dots)$$

converge (0)-weakly as  $m \rightarrow 0$ , then the lower derivatives converge in the norm of those spaces in which they are contained by Theorems 1-3.

In proving this theorem we use a theorem of Ya. B. Rutitskii (5) on the complete continuity of integral operators acting in Orlicz spaces.

Theorems 1-5 completely generalize the embedding theorems of S. L. Sobolev ((1), p. 78) and the theorems on complete continuity of the embedding operator for the spaces  $W_\alpha^{(l)}$  ((1), pp. 91, 94). With their help one proves:

**Theorem 6.** Let  $\{h_{\alpha_1, \dots, \alpha_n}\}$  ( $\sum \alpha_i \leq l-1$ ) be some system of distributive functionals, bounded in  $C$  or in  $L_{M_1}^*(\omega^{(s)})$  (depending on in which of these spaces the range of the embedding operator lies), and let for every polynomial of degree not exceeding  $(l-1)$  at least one of them be different from zero. Then the norm

$$\|u\|_{F_M^{(l)}} = \sum_{\sum \alpha_i \leq l-1} |h_{\alpha_1, \dots, \alpha_n} u| + \|D^{(l)}u\|_M \quad (4)$$

is equivalent to the norm (2).

**2. Theorem 7.** Let the integrand in the integral

$$j(u, p) = \int_{\Omega} f(x_1, \dots, x_n; u_1(x), \dots, u_m(x); p_1(x), \dots, p_k(x)) d\Omega$$

have the following properties:

- 1)  $f(x, u, p) \geq 0$  for all  $x \in \Omega$  and arbitrary  $u, p$ .
- 2)  $f$  is continuous, together with the partial derivatives  $f_{p_s} = \partial f / \partial p_s$  ( $s = 1, 2, \dots, k$ ), everywhere where it is defined.

3) The function

$$\mathcal{E}(x, u, p, \bar{p}) = f(x, u, p) - f(x, u, \bar{p}) - \sum_{s=1}^k (p_s - \bar{p}_s) f_{p_s}(x, u, \bar{p}) \geq 0$$

everywhere where it is defined.

Then, if  $u_r^{(i)}(x) \rightarrow u_r^{(0)}(x)$  ( $r = 1, 2, \dots, m$ ) in the norm of the space  $L_M^*$ , and  $p_s^{(i)}(x) \rightarrow p_s^{(0)}(x)$  ( $s = 1, 2, \dots, k$ ) ( $o$ )-weakly, then

$$\lim_{i \rightarrow \infty} j(u^{(i)}, p^{(i)}) \geq j(u^{(0)}, p^{(0)}).$$

(If  $j(u^{(0)}, p^{(0)}) = \infty$ , then  $\lim_{i \rightarrow \infty} j(u^{(i)}, p^{(i)}) = \infty$ .)

This theorem, in the case where  $M(u) = |u|^\alpha / \alpha$  ( $\alpha > 1$ ), was proved by V. I. Kazimirov <sup>(6)</sup>. The extension of the proof to the general case presents no essential difficulties.

Let  $\Gamma$  be the boundary of the domain  $\Omega$ , and let the dimension of  $\Gamma$  satisfy the conditions of the embedding theorems. In the integral

$$j(\mathbf{u}) = \int_{\Omega} f(x_1, \dots, x_n; u_1(x), \dots, u_R(x); \dots \dots \frac{\partial^k u_i}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \dots; \dots \frac{\partial^l u_i}{\partial x_1^{l_1} \dots \partial x_n^{l_n}}; \dots) d\Omega \quad (5)$$

we impose on the integrand the condition

$$f \geq c_1 M(D^{(l)} \mathbf{u}) \quad \left( c_1 > 0 \text{ is a constant, } D^{(l)} \mathbf{u} = \sum_{i=1}^R D^{(l)} u_i \right) \quad (6)$$

for all  $x \in \Omega$  and arbitrary  $D^{(k)} \mathbf{u}$  ( $k = 0, 1, \dots, l-1$ ). Then, if  $j(\mathbf{u}) < +\infty$ , then  $\mathbf{u} \in F_M^{(l)}$  ( $F_M^{(l)}$  is the space of vector-functions  $\mathbf{u} = (u_1(x), \dots, \dots, u_R(x))$ ,  $u_i \in F_M^{(l)}$ ). We shall say that the vector-function  $\mathbf{u}$  belongs to the class  $\Phi$  of admissible functions if  $j(\mathbf{u}) < +\infty$  and

$$\left. \frac{\partial^k u_i}{\partial x_1^{k_1} \dots \partial x_n^{k_n}} \right|_{\Gamma} = 0 \quad (i = 1, 2, \dots, R; k = 0, 1, \dots, l-1). \quad (7)$$

In the class  $\Phi$  we introduce a norm by the method (4), taking

$$h_{\alpha_1, \dots, \alpha_n} u = \int_{\Gamma} \frac{\partial^\alpha u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} dx \quad (\alpha = 0, 1, \dots, l-1; \sum \alpha_i = \alpha). \quad (8)$$

**Theorem 8.** Let the integrand  $f$  in the integral (5) satisfy condition (6) and conditions 2), 3) of Theorem 7 (the highest derivatives of the vector-function  $\mathbf{u}$  are taken as the functions  $p_s(x)$ ). Then there exists a vector-function  $\mathbf{u}_0 \in \Phi$  such that

$$j(\mathbf{u}_0) = \inf_{\mathbf{u} \in \Phi} j(\mathbf{u}).$$

Moreover, some minimizing sequence converges to  $\mathbf{u}_0$  (0)-weakly in  $F_M^{(l)}$ , i.e. the highest derivatives of this sequence converge (0)-weakly in the space  $L_M^*$ . The solution of the problem under consideration is unique if

$$\sum_{i,k=1}^R \sum_{r,s=0}^l \frac{\partial^2 f}{\partial u_{r_1, \dots, r_n}^{i,r} \partial u_{s_1, \dots, s_n}^{k,s}} t_{r_1, \dots, r_n}^{i,r} t_{s_1, \dots, s_n}^{k,s} > 0 \quad \left( \sum |t_{\beta_1, \dots, \beta_n}^{\alpha, \beta}|^2 > 0 \right). \quad (9)$$

Any minimizing sequence converges in this case to  $\mathbf{u}_0$  (0)-weakly in  $\mathbf{F}_M^{(1)}$ .

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Leningrad State  
Pedagogical Institute  
named after A. I. Herzen

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## CORRECTION

In my article published in DAN, vol. 120, No. 3, 1958 (I. V. Gel' man, "On a Certain Nonlinear Operator" ), in the list of references one should read: 3. I. V. Gel' man, DAN, 122, No. 4 (1958).

*I. V. Gel' man*

*Note: Figure translations are in progress. See original paper for figures.*

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