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Abstract

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MATHEMATICS

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STRUCTURAL CHARACTERIZATION OF CERTAIN CLASSES OF ALGEBRAS

Mac Lane ⁽¹⁾ found conditions that a general category must satisfy in order for it to be isomorphic to the category of all Abelian semigroups with zero. In the more special theory of categories of structures, it is natural, alongside general isomorphism, to consider also a more special kind of isomorphism—structural equivalence. The main aim of the present note is to find conditions that a category of structures must satisfy in order for it to be structurally equivalent to a subclass of a class of algebras of a fixed type, closed under multiplication and containing subalgebras of its algebras. Below we use throughout the terminology and results of the note ⁽²⁾.

1. Let L be some subcategory of a category of structures K in the sense of ⁽²⁾, and let $\mathfrak{A} \in K$. A system of elements of the structure \mathfrak{A} will be called L -free if every mapping of this system into an arbitrary L -structure \mathfrak{B} can be extended to a homomorphism of \mathfrak{A} into \mathfrak{B} . The structure \mathfrak{A} will be called L -free if $\mathfrak{A} \in L$ and if there exists in \mathfrak{A} an L -dense L -free system of elements. It follows from this that L -free structures possessing L -dense L -free systems of the same cardinality are isomorphic. We also note that if L is a regular, bounded, multiplicatively closed class of structures, then in L there exist L -free structures with L -dense L -free systems of arbitrary cardinality.

We shall agree to call a subcategory L of a category of structures K homomorphically closed in K if the homomorphic image of an L -structure in a K -structure is an L -substructure of the latter. We shall say that a system of elements S of a structure \mathfrak{A} is an L -generating system for \mathfrak{A} if \mathfrak{A} contains no L -substructures containing S and distinct from \mathfrak{A} .

In what follows we consider only categories of structures with strong substructures, i.e. only those categories in which a homomorphism into a substructure is a homomorphism into a structure.

Theorem 1. *Let the category of structures L be homomorphically closed in itself and contain L -free structures of arbitrary cardinality. Then the category L is bounded, regular, the intersection of any system of L -substructures of an L -structure is either empty or is an L -substructure, every generating system of elements of an L -structure is dense in it, and the complete preimage of an*

L -substructure of an L -structure \mathfrak{A} under a homomorphic mapping onto an L -structure \mathfrak{B} is an L -substructure in \mathfrak{B} .

We shall agree to call a structure \mathfrak{A} L -freely cyclic if \mathfrak{A} has an L -dense L -free element and $\mathfrak{A} \in L$. Then, if the category L contains an L -freely cyclic structure, every L -free structure will be an L -free composition of L -freely cyclic ones.

Let us note the following cases when dense systems are generating. The totality of elements of the canonical image of a K -structure in its L -replica is L -generating in the replica. If a K -structure \mathfrak{A} contains an L -free system of elements S , then \mathfrak{A} can contain at most one L -substructure in which S is an L -dense system. Therefore

an L -free structure is L -generated by its L -dense L -free system.

2. By analogy with group theory, we shall call a system $\{M_\alpha\}$ of subsets of some set **local** if every finite subset of elements from $\bigcup M_\alpha$ lies in a suitable subset M_β . We shall call a category of structures K **additive** if the union of any local system of K -substructures of an arbitrary K -structure is a K -substructure. We shall call a K -dense system S of elements of some structure \mathfrak{A} **K -finitarily dense** if every element of \mathfrak{A} lies in a K -substructure containing a suitable finite part of S as a dense system.

Theorem 2. *Every homomorphically self-closed class of structures K that contains K -free structures with K -free K -finitarily dense systems of any prescribed cardinality is additive. In regular additive classes K , K -free K -dense systems of elements of structures are K -finitarily dense.*

From the isomorphism of K -free structures with K -free K -dense systems of the same cardinality it follows that if there exists a K -free structure with a K -free K -finitarily dense system of cardinality \mathfrak{m} , then every K -free K -dense system of cardinality \mathfrak{m} will be finitarily dense.

3. Suppose that in the category of structures L there are L -free structures with an L -dense L -free set consisting of any fixed finite number of elements. Denote by the symbol V_n an L -free structure, and by the symbols $v_{n\alpha}$ ($\alpha \in \Gamma_n$; $1, 2, \dots, n \in \Gamma_n$) its elements, among which let v_{n1}, \dots, v_{nn} form an L -dense L -free system in V_n ($n = 1, 2, \dots$). We now define on each L -structure \mathfrak{A} a series of operations $\Phi_{n\alpha}(x_1, \dots, x_n)$ ($\alpha \in \Gamma_n$; $n = 1, 2, \dots$) as follows. Let a_1, \dots, a_n be a sequence of elements of \mathfrak{A} . By assumption there exists a unique homomorphism σ of the structure V_n into \mathfrak{A} for which $v_{ni}^\sigma = a_i$ ($i = 1, \dots, n$). By definition, we put $\Phi_{n\alpha}(a_1, \dots, a_n) = v_{n\alpha}^\sigma$. It is clear from this that if two structures \mathfrak{A} and \mathfrak{B} are given on one and the same set M , then the value of the expression $\Phi_{n\alpha}(a_1, \dots, a_n)$ ($a_1, \dots, a_n \in M$) in \mathfrak{A} may be different from that in \mathfrak{B} . However, it follows from the definition that if \mathfrak{B} is an L -substructure of \mathfrak{A} and a_1, \dots, a_n belong to \mathfrak{B} , then the value of $\Phi_{n\alpha}(a_1, \dots, a_n)$ in \mathfrak{A} and in \mathfrak{B} is one and the same, i.e. the operations Φ are **stable** under passage to L -over- and L -substructures. Further, the operations $\Phi_{n\alpha}$ are **preserved** under homomorphisms, i.e. if

σ is a homomorphism of an L -structure \mathfrak{A} into an L -structure \mathfrak{B} , then

$$(\Phi_{n\alpha}(a_1, \dots, a_n))^\sigma = \Phi_{n\alpha}(a_1^\sigma, \dots, a_n^\sigma) \quad (a_1, \dots, a_n \in \mathfrak{A}).$$

It follows from this that the Φ -operations are invariant under passage to direct products.

Theorem 3. *Let a homomorphically self-closed category of structures L contain L -free structures with L -finitarily dense L -free systems of any cardinality. Under these conditions, a subset \mathfrak{B} of the elements of an L -structure \mathfrak{A} is an L -substructure in \mathfrak{A} if and only if it is closed with respect to all operations Φ .*

Here Φ -closedness of \mathfrak{B} means that for all a_1, \dots, a_n from \mathfrak{B} we have $\Phi(a_1, \dots, a_n) \in \mathfrak{B}$.

4. We shall call a category of structures L a category with **divisible homomorphisms** if, for any homomorphism σ of an L -structure \mathfrak{A} onto an L -structure \mathfrak{B} , and any mapping φ of the structure \mathfrak{B} into an L -structure \mathfrak{C} , it follows from the fact that $\sigma\varphi$ is a homomorphism that φ is a homomorphism. It is easy to see that every class of algebras is a category with divisible homomorphisms.

Theorem 4. *Let a category L with divisible homomorphisms contain L -free structures with L -finitarily dense L -free systems of arbitrary cardinality. Then every mapping of an L -structure \mathfrak{A} into an L -structure \mathfrak{B} that preserves the Φ -operations is a homomorphism from \mathfrak{A} to \mathfrak{B} .*

Categories K_1 and K_2 are called isomorphic ⁽⁴⁾ if it is possible to establish a one-to-one correspondence ψ between the elements (homomorphisms) of K_1 and K_2 , which is an isomorphism when K_1 and K_2 are regarded as partial semigroups. In the case of categories of structures this means that, from a K_1 -structure with one underlying set, the rule ψ must allow one to construct a K_2 -structure, in general with another underlying set, and from each homomorphism of K_1 -structures to construct a homomorphism of the corresponding K_2 -structures having the appropriate additional properties. For us the following, stronger concept of **structural equivalence** of categories will be important. We shall call the category of structures K_1 structurally equivalent to the category of structures K_2 if a rule ψ is given that makes it possible, for each K_1 -structure, to construct uniquely a K_2 -structure with the same underlying set, if, moreover, every homomorphism of a K_1 -structure \mathfrak{A} into a K_1 -structure \mathfrak{B} is a homomorphism \mathfrak{A}^ψ into \mathfrak{B}^ψ , and if there exists an inverse rule with the corresponding properties. From Theorem 4 the following corollary is now obtained immediately.

Corollary. *Every category of structures K with divisible homomorphisms, containing K -free structures with K -finitarily dense K -free systems of arbitrary cardinality, is structurally equivalent to some subcategory of the category of all algebras of the appropriate type.*

Indeed, it was indicated above how to convert K -structures into algebras of type $\{\Phi_{m_\alpha}\}$. Denote by K_2 the class of all those algebras which can be obtained from K -structures in this way. Theorem 4 shows that the correspondence between the objects of K and K_2 is one-to-one and satisfies the condition that homomorphisms coincide.

5. We shall call a subcategory L of a category of structures K **quasi-free in K** if L contains the unit structure, is multiplicatively closed in K , and the K -substructures of L -structures are L -structures. We shall call a subcategory L **free in K** if it is quasi-free and homomorphically closed in K . According to Birkhoff's theorem, every free subcategory of the category of all algebras of a fixed type is a class of algebras characterized by a system of identities, i.e. a primitive class in the sense of (3). A special kind of quasi-free subcategories will be the quasi-primitive classes of algebras (3). If the base category is regular and bounded, then every quasi-free subcategory of it will be R -complete. In particular, every quasi-free subclass of the class of all algebras of a fixed type is R -complete. These subclasses can also be characterized by their purely structural properties.

Theorem 5. *In order that a category of structures K be structurally equivalent to a quasi-free subclass of the category of all algebras of some fixed type, it is necessary and sufficient that K contain the unit structure and be regular, bounded, additive, multiplicatively and homomorphically closed in itself.*

The necessity follows immediately from elementary properties of algebras. The sufficiency follows from the totality of the preceding theorems.

6. A class L_1 of algebras with fundamental operations $f_\alpha(x_1, \dots, x_{m(\alpha)})$ ($\alpha \in \Gamma_1$) is called **rationally equivalent** to a class L_2 of algebras with fundamental operations $g_\beta(x_1, \dots, x_{n(\beta)})$ ($\beta \in \Gamma_2$), if there exist such L_2 -polynomials $F_\alpha(x_1, \dots, x_{m(\alpha)})$ and such L_1 -polynomials $G_\beta(x_1, \dots, x_{n(\beta)})$ ($\alpha \in \Gamma_1, \beta \in \Gamma_2$) that every L_1 -algebra, considered with respect to the G -operations, is an L_2 -algebra, and every L_2 -algebra with respect to the F -operations is ...

L_1 -algebra and the indicated correspondence are involutive⁵. Rational equivalence is in general distinct from structural equivalence, but may also coincide with it. Let us note the simplest case of this:

Theorem 6. *If quasivariety subclasses of the classes of all algebras of generally distinct fixed types are structurally equivalent, then they are also rationally equivalent.*

Let the given subclasses be L_1, L_2 , and let $f_\alpha(x_1, \dots, x_m)$ be one of the basic operations of the class L_1 . Consider an L_1 -free algebra V with L_1 -free generators v_1, \dots, v_m . From the structural equivalence of L_1, L_2 it follows that V will also be an L_2 -free structure with L_2 -free generators v_1, \dots, v_m . Therefore the element $f_\alpha(v_1, \dots, v_m)$ of the algebra V must be representable in the form of some L_2 -polynomial $F_\alpha(v_1, \dots, v_m)$. From the equality $f_\alpha(v_1, \dots, v_m) = F_\alpha(v_1, \dots, v_m)$

in V it follows that in L_1 the identity $f_\alpha(x_1, \dots, x_m) = F_\alpha(x_1, \dots, x_m)$ holds. Similarly, we obtain that for each basic L_2 -operation $g_\beta(x_1, \dots, x_n)$ there exists an L_1 -polynomial $G_\beta(x_1, \dots, x_n)$ for which in L_1 the identity $g_\beta(x_1, \dots, x_n) = G_\beta(x_1, \dots, x_n)$ holds. Thus, the classes L_1, L_2 are rationally equivalent.

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