

On the Calculation of the Horizontal Turbulent Diffusion of Impurity Patches in the Sea

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Abstract

Full Text

Geophysics

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On the Calculation of the Horizontal Turbulent Diffusion of Impurity Patches in the Sea

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The methods used in oceanography for calculating the horizontal turbulent diffusion of impurity patches in the sea are imperfect. The main shortcoming of such calculations is the assumption that the coefficient of horizontal turbulent diffusion is constant in the equation describing the diffusion of a patch ⁽¹⁾. According to available investigations ^(2, 3), however, the diffusion coefficient in a turbulent flow depends essentially on the scale of the phenomenon and is described by the “four-thirds power” law:

$$K = cl^{4/3}, \quad (1)$$

where l is the scale of the phenomenon, and c is a certain constant.

Thus, a calculation of the horizontal turbulent diffusion of an impurity patch in the sea with a constant diffusion coefficient can give only a first, rough approximation to the actually occurring process. In reality, when a patch has small dimensions, it diffuses slowly, and only as the patch increases does the rate of its diffusion also increase.

The dependence of the coefficient of turbulent diffusion on the time of diffusion is given by Taylor’s well-known formula ⁽⁴⁾. However, the complicated functional dependence of the diffusion coefficient on difficult-to-measure characteristics of turbulence (the Lagrangian correlation coefficient) makes it possible only in particular cases to estimate the dispersion of a patch as a function of time. It does not seem possible to use the diffusion coefficient given by Taylor’s formula for calculations of the distribution of impurity concentration in a diffusing patch, in view of the mathematical difficulties that arise.

Experiments carried out by Richardson and Stommel ^(5, 6), as well as by the author ⁽⁷⁾, showed that the diffusion coefficient of particles on the water surface is described with sufficient accuracy by the “four-thirds power” law. Consequently, it may be asserted that the peripheral zones of an impurity patch, being the most distant from one another, are “stretched out” by turbulent eddies considerably faster than the central parts of the patch.

The considerations set forth suggest that the equation of horizontal turbulent

diffusion of an impurity patch in the sea should most naturally be written in the form

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial x} \left(c\sqrt{x^2 + y^2}^{4/3} \frac{\partial q}{\partial x} \right) + \frac{\partial}{\partial y} \left(c\sqrt{x^2 + y^2}^{4/3} \frac{\partial q}{\partial y} \right), \quad (2)$$

where q is the concentration of the diffusing impurity; x and y are the Cartesian coordinates of the point of observation (with the origin of the system at the center of the patch); c is the constant in the “four-thirds power” law, close to $0.01 \text{ cm}^{2/3} \cdot \text{sec}^{-1}$ (7, 8).

It should be noted that equation (2) does not take into account the diffusion of the impurity in the vertical direction, which, according to the available data, is considerably weaker than horizontal diffusion, and for an impurity having a floating buoyancy, and is even absent altogether. For some simplification of the subsequent calculations, we shall assume that there is no mean current carrying the impurity patch as a whole.

Equation (2), evidently, should be written in polar coordinates. Under the condition that the process is independent of the angle φ (horizontal isotropy of the process), we obtain

$$\frac{\partial q}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[K(r)r \frac{\partial q}{\partial r} \right], \quad (3)$$

where $K(r) = cr^{4/3}$; r is the radius vector, or, differentiating,

$$\frac{\partial q}{\partial t} = c \left[\frac{7}{3} r^{1/3} \frac{\partial q}{\partial r} + r^{4/3} \frac{\partial^2 q}{\partial r^2} \right]. \quad (4)$$

Making the change of variables according to the formula

$$r^{1/3} = \rho, \quad (5)$$

we reduce equation (4) to the form

$$\frac{\partial q}{\partial t} = \frac{c}{9} \left[\frac{5}{\rho} \frac{\partial q}{\partial \rho} + \frac{\partial^2 q}{\partial \rho^2} \right]. \quad (6)$$

We seek a solution of equation (6) in the form

$$q(\rho, t) = at^n e^{-\rho^2/4 \frac{c}{9} t}, \quad (7)$$

where a and n are constants to be determined.

Differentiating (7) and substituting into (6), we find that $n = -3$. Consequently, the solution of equation (6) has the form

$$q(\rho, t) = \frac{a}{t^3} e^{-\rho^2/4 \frac{c}{9} t}. \quad (8)$$

The constant a is determined from the condition of non-destruction of the substance in the diffusion process:

$$Q = \int_0^\infty 2\pi r q(r) dr, \quad (9)$$

where Q is the total amount of the diffusing substance.

Substituting (8) into (9) and integrating, we find

$$a = \frac{Q}{6\pi \left(4 \frac{c}{9}\right)^3}. \quad (10)$$

Thus, the solution of equation (3) has the form

$$q(r, t) = \frac{Q}{6\pi \left(4 \frac{c}{9}\right)^3 t^3} e^{-r^2/4 \frac{c}{9} t}. \quad (11)$$

Expression (11) gives the concentration of the impurity caused by diffusion from a point source of strength Q , placed at time $t = 0$ at the origin of coordinates.

An essential difference between the expression (11) obtained and the known solution of the diffusion equation with a constant coefficient is the rapid fall of the impurity concentration at the center of the patch that follows from (11) (proportional to the third power of time). As is known, according to the solution of the two-dimensional diffusion equation with a constant coefficient, the con-

centration of the impurity at the center of the patch decreases proportionally to the first power of t . Even the three-dimensional diffusion equation with a constant coefficient (i.e., under the assumption of the presence of intense diffusion in the vertical direction) gives a rate of decrease of the impurity concentration at the center of the patch proportional only to $t^{3/2}$.

The decrease in concentration with distance from the center of the patch occurs, according to expression (11), more slowly (the exponent is proportional to $r^{4/3}$) than follows from the solution of the diffusion equation with a constant coefficient (the exponent is proportional to r^2).

Fig. 1. Change with time of the concentration of a diffusing impurity at the center of the patch according to the experimental data of Nanniti and Okubo. 1 –dependence given by formula (11); 2, 3 –dependences following from solutions

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Figure 1: Fig. 1. Change with time of the concentration of a diffusing impurity at the center of the patch according to the experimental data of Nanniti and Okubo. 1 –dependence given by formula (11); 2, 3 –dependences following from solutions of the diffusion equations with constant coefficients (2 –two-dimensional case, 3 –three-dimensional).

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It is most convenient to compare the formula obtained (11) with observational data on the diffusion of impurity patches in the sea by measuring the impurity concentration at the center of the patch. We were able to find only one paper, by Nanniti and Okubo ⁽⁹⁾, in which such data are given. The paper describes observations of the horizontal diffusion of a patch of dye released into the sea near the shores of Japan. Nanniti and Okubo give the following data on the dye concentration at the center of the patch at four different times (time is counted from the moment of dye release):

	40	104	182	252
t , min.	40	104	182	252
q (in units of 10^{-7} of the initial concentration)	10000	850	250	48

The data given are presented in logarithmic coordinates in Fig. 1. The same figure shows the straight lines for the change of impurity concentration at the center of the patch with time, corresponding to expression (11) and to the solutions of the diffusion equation with a constant coefficient (for the two-dimensional and three-dimensional cases). As is seen from the drawing, the points confirm the dependence given by formula (11), and deviate sharply from the straight lines following from solutions of the diffusion equation with any constant coefficient.

Such agreement of the experimental data with the theoretical dependence gives confidence that the equation of horizontal turbulent diffusion (3), with the diffusion coefficient specified by the “four-thirds power” law, corresponds more closely to the process occurring in nature than do diffusion equations with constant coefficients, and, consequently, an equation of type (3) can be successfully

applied to the calculation of problems on the horizontal turbulent diffusion of impurity patches in the sea.

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