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Abstract

Full Text

APPROXIMATION OF DIFFERENTIABLE FUNCTIONS BY VALLEE-POUSSIN SUMS

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We shall consider continuous functions $f(x)$ of period 2π. Let

$$s_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \quad (n = 0, 1, 2, \ldots),$$

$$\sigma_n(f,x) = \frac{1}{n}\sum_{k=0}^{n-1} s_k(f,x) \quad (n = 1, 2, \ldots)$$

be the partial sums of the Fourier series and the Fejér sums of the function $f(x)$. Vallée-Poussin ([2,3]) first considered the polynomials

$$v_{n,m}(f,x) = \frac{1}{m}\sum_{k=n-m}^{n-1} s_k(f,x) = \frac{n}{m}\sigma_n(f,x) - \frac{n-m}{m}\sigma_{n-m}(f,x) \tag{1}$$

$$[(m = 1, 2, \ldots, n;\ n = 1, 2, \ldots)],$$

which received the name of Vallée-Poussin sums, and there also indicated the formula

$$|f(x) - v_{n,m}(f,x)| \le 2\frac{n}{m}E_{n-m}(f), \tag{2}$$

where $E_k(f)$, $k = 0, 1, 2, \ldots$, is the best approximation of the function $f(x)$ by polynomials of order $k-1$.

Let W^r be the class of functions $f(x)$ for which the derivative $f^{(r-1)}(x)$ is absolutely continuous and the inequality $|f^{(r)}(x)| \le 1$ holds almost everywhere, and let $\overline{W}^r$ be the class of functions conjugate to functions of W^r, $r = 1, 2, \ldots$. In this paper we determine the asymptotic behavior as $n \to \infty$ of the quantity

$$V_{n,m}(\mathfrak{M}) = \sup_{f\in\mathfrak{M}} \|f(x) - v_{n,m}(f,x)\|_C$$

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for $\mathfrak{M} = W^r$ and $\mathfrak{M} = \overline{W}^r$, under the assumption that $\lim \frac{m}{n}$ exists and is equal to θ, $0 \leq \theta \leq 1$.

For Fourier sums ($m = 1$), Vallée-Poussin sums close to them (the case $m = o(n)$), and Fejér sums ($m = n$), the asymptotic behavior of $V_{n,m}(W^r)$ and $V_{n,m}(\overline{W}^r)$ is known.

If $m = o(n)$, then

$$V_{n,m}(W^r) = \frac{4}{\pi^2}\frac{1}{n^r}\log\frac{n}{m} + O\left(\frac{1}{n^r}\right), \tag{3}$$

$$V_{n,m}(\overline{W}^r) = \frac{4}{\pi^2}\frac{1}{n^r}\log\frac{n}{m} + O\left(\frac{1}{n^r}\right). \tag{4}$$

For Fourier sums, formula (3) was obtained by A. N. Kolmogorov ([4]), and formula (4) by S. M. Nikol' skii ([8,9]); for the case $m = o(n)$ these formulas were obtained by A. F. Timan ([11,12]).

For approximations by Fejér sums, S. M. Nikol' skii ([5,7,9]) obtained the following asymptotic formulas:

$$V_{n,n}(W^1) = \frac{2}{\pi}\frac{1}{n}\log n + O\left(\frac{1}{n}\right)$$

and for $r > 1$

$$V_{n,n}(W^r) = \frac{c_r}{n} + O\left(\frac{1}{n^r}\right), \qquad V_{n,n}(\overline{W}^r) = \frac{\overline{c}_r}{n} + O\left(\frac{1}{n^r}\right),$$

where

$$c_r = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k(r-1)}}{(2k+1)^r}, \qquad \overline{c}_r = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{kr}}{(2k+1)^r}.$$

S. B. Stechkin communicated to me the formula*

$$V_{n,n}(\overline{W}^1) = \frac{2}{\pi n}\int_0^{\infty}\left|\int_u^{\infty}\frac{\sin t}{t^2}\,dt\right|du + O\left(\frac{1}{n^2}\right). \tag{5}$$

It is also known that in the case $0 < \theta < 1$

$$V_{n,m}(W^r) = O\left(\frac{1}{n^r}\right), \qquad V_{n,m}(\overline{W}^r) = O\left(\frac{1}{n^r}\right).$$

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These relations follow from inequality (2) and theorems on the order of best approximations of functions from W^r and $\overline{W}^r$.

The method used in this paper gives the asymptotic behavior of $V_{n,m}(W^r)$ and $V_{n,m}(\overline{W}^r)$ for $m = 1, 2, \dots, n-1$, for all $r = 1, 2, \dots$; for $m = n$ (Fejér sums), only for $r = 1$.

Theorem. For $V_{n,m}(W^r)$ and $V_{n,m}(\overline{W}^r)$ the following asymptotic formulas hold:

1. If $\theta = 0$, formulas (3) and (4).
2. If $0 < \theta < 1$, then

$$V_{n,m}(W^r) = c(r,\theta)\frac{1}{n^r} + O\left(\frac{1}{n^{r+1}}\right) + O\left(\frac{\varepsilon_n}{n^r}\right); \tag{6}$$

$$V_{n,m}(\overline{W}^r) = \overline{c}(r,\theta)\frac{1}{n^r} + O\left(\frac{1}{n^{r+1}}\right) + O\left(\frac{\varepsilon_n}{n^r}\right), \tag{7}$$

where

$$\left.\begin{matrix} c(r,\theta) \\ \overline{c}(r,\theta) \end{matrix}\right\} = \frac{2}{\pi\theta}\int_0^\infty \left| \int_{u_r}^\infty \cdots \int_{u_1}^\infty \left\{ \begin{matrix} \cos \\ \sin \end{matrix}\ u - \frac{\begin{matrix} \cos \\ \sin \end{matrix}\ (1-\theta)u}{} \right\} u^{-2}\, du \dots du_{r-1} \right| du_r,$$

$$\varepsilon_n = \left|\frac{m}{n} - \theta\right| \log\frac{1}{|m/n - \theta|} \quad \text{for } \frac{m}{n} \neq \theta, \qquad \varepsilon_n = 0 \quad \text{for } \frac{m}{n} = \theta.$$

3. If $\theta = 1$, then for $r > 1$ the case $n - m$ fixed and $n - m \to \infty$ are considered separately.

The case $n - m = p$ fixed, $p \geq 1$:

$$V_{n,m}(W^r) = c(r,p)\left[\frac{1}{n} + \frac{p}{n^2} + \dots + \frac{p^{r-2}}{n^{r-1}}\right] + O\left(\frac{1}{n^r}\right); \tag{8}$$

$$V_{n,m}(\overline{W}^r) = \overline{c}(r,p)\left[\frac{1}{n} + \frac{p}{n^2} + \dots + \frac{p^{r-2}}{n^{r-1}}\right] + O\left(\frac{1}{n^r}\right), \tag{9}$$

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where

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$$\left.\begin{matrix} c(r,p) \\ \bar{c}(r,p) \end{matrix}\right\} = \frac{2}{\pi} \sup_f \left| \int_0^\infty f^{(r)}(u_r) \int_{u_r}^\infty \cdots \int_{u_1}^\infty \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} pu\, u^{-2}\, du \dots du_r \right|$$

and the upper bound is taken over functions $f \in W^r$, even for $c(r,p)$ and odd for $\bar{c}(r,p)$.

The case $n-m \to \infty$:

$$V_{n,m}(W^r) = c(r,\infty) \left[\frac{1}{n(n-m)^{r-1}} + \frac{1}{n^2(n-m)^{r-2}} + \cdots \right.$$
$$\left. \cdots + \frac{1}{n^{r-1}(n-m)} \right] + O\left(\frac{1}{n^r}\right) + O\left(\frac{1}{n(n-m)^r}\right); \tag{10}$$

$$V_{n,m}(\overline{W}^r) = \bar{c}(r,\infty) \left[\frac{1}{n(n-m)^{r-1}} + \frac{1}{n^2(n-m)^{r-2}} + \cdots \right.$$
$$\left. \cdots + \frac{1}{n^{r-1}(n-m)} \right] + O\left(\frac{1}{n^r}\right) + O\left(\frac{1}{n(n-m)^r}\right), \tag{11}$$

where

$$\left.\begin{matrix} c(r,\infty) \\ \bar{c}(r,\infty) \end{matrix}\right\} = \frac{2}{\pi} \int_0^\infty \left| \int_{u_r}^\infty \cdots \int_{u_1}^\infty \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} u\, u^{-2}\, du \dots du_{r-1} \right| du_r.$$

For $r = 1$

$$V_{n,m}(W^1) = \frac{2}{\pi} \frac{1}{n} \log \frac{n}{n-m+1} + O\left(\frac{1}{n}\right). \tag{12}$$

In the case $n-m = p$ fixed, $p \geq 0$:

$$V_{n,m}(\overline{W}^1) = \bar{c}(1,p) \frac{1}{n} + O\left(\frac{1}{n}\sqrt{\frac{\log n}{n}}\right), \tag{13}$$

where

$$\bar{c}(1,p) = \frac{2}{\pi} \int_0^\infty \left| \int_u^\infty \frac{\sin t}{t^2}\, dt \right| du + pV_{p,p}(\overline{W}^1).$$

In the case $n-m \to \infty$

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$$V_{n,m}(\overline{W}^1) = \bar{c}(1,\infty)\frac{1}{n} + O\left(\frac{1}{n}\sqrt{\frac{n-m}{n}}\log\frac{n}{n-m}\right) + O\left(\frac{1}{n(n-m)}\right), \quad (14)$$

where

$$\bar{c}(1,\infty) = \frac{4}{\pi}\int_0^\infty \left|\int_u^\infty \frac{\sin t}{t^2}\,dt\right| du.$$

For comparison we give the asymptotic formulas obtained by S. M. Nikol' skii ([6]) for the norms of the de la Vallée Poussin sums,

$$V_{n,m} = \frac{4}{\pi^2}\log\frac{n}{m} + O(1) \quad \text{for } \theta = 0; \quad (15)$$

$$V_{n,m} = \frac{2}{\pi}\int_0^\infty \frac{|\cos u - \cos(1-\theta)u|}{u^2}\,du + O(\varepsilon_n) \quad \text{for } 0 < \theta \le 1. \quad (16)$$

In ([6]) the order of decrease of the remainder term in (16) is not indicated; this refinement is easily obtained from the work of S. B. Stechkin ([10]).

For the proof of the theorem one uses the representation of de la Vallée Poussin sums by Fejér sums (([3]), see also ([1])), from which it follows that

$$f(x) - v_{n,m}(f,x) = \frac{1}{\pi m}\int_{-\infty}^{\infty}[f(x+u) - f(x)]\,\frac{\cos nu - \cos(n-m)u}{u^2}\,du.$$

In determining $V_{n,m}(W^r)$ it suffices to consider only the deviation:

to zero for even functions. In this case we have

$$f(0) - v_{n,m}(f,0) = -\frac{2}{\pi m}\int_0^\infty f^{(r)}(u_r)\int_{u_r}^\infty \cdots \int_{u_1}^\infty \frac{\cos nu - \cos(n-m)u}{u^2}\,du\ldots du_r. \quad (17)$$

From this (8) follows immediately, as well as the inequality

$$V_{n,m}(W^r) \leqslant \frac{2}{\pi m}\int_0^\infty \left|\int_{u_r}^\infty \cdots \int_{u_1}^\infty \frac{\cos nu - \cos(n-m)u}{u^2}\,du\ldots du_{r-1}\right| du_r. \quad (18)$$

Next one constructs a function showing that

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$$V_{n,m}(W^r) = \frac{2}{\pi m}\int_0^\infty \left| \int_{u_r}^\infty \cdots \int_{u_1}^\infty \frac{\cos nu - \cos(n-m)u}{u^2}\, du \ldots du_{r-1} \right| du_r + O\left(\frac{1}{m(n-m)^r}\right). \tag{19}$$

If $\bar{f}(x)$ is the function conjugate to $f(x)$, then

$$V_{n,m}(\overline{W^r}) = \sup_{f\in W^r} \left|\bar{f}(0) - v_{n,m}(\bar{f}, 0)\right|,$$

and by analogous arguments we obtain

$$V_{n,m}(\overline{W^r}) = \frac{2}{\pi m}\int_0^\infty \left| \int_{u_r}^\infty \cdots \int_{u_1}^\infty \frac{\sin nu - \sin(n-m)u}{u^2}\, du \ldots du_{r-1} \right| du_r + O\left(\frac{1}{m(n-m)^r}\right). \tag{20}$$

From (19), (20) we obtain (10), (11), and, since for small α

$$\int_0^\infty \left| \int_{u_r}^\infty \cdots \int_{u_1}^\infty \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} u - \left\{ \begin{array}{c} \cos \\ \sin \end{array} \right\} (1+\alpha)u \right| u^{-2} du \ldots du_{r-1} \Bigg| du_r = \frac{2}{\pi}|\alpha| \log\frac{1}{|\alpha|} + O(|\alpha|),$$

also formulas (3), (4), (6), (7).

To obtain formulas (12) and (13) the arguments given above are not suitable. For $\theta = 1$ we have

$$V_{n,m}(W^1) = \frac{n}{m} V_{n,n}(W^1) - \frac{n-m}{m} V_{n-m,n-m}(W^1) + O\left(\frac{1}{m}\right); \tag{21}$$

$$V_{n,m}(\overline{W^1}) = \frac{n}{m} V_{n,n}(\overline{W^1}) + \frac{n-m}{m} V_{n-m,n-m}(\overline{W^1}) + O\left(\frac{1}{m}\sqrt{\frac{n-m}{n}\log\frac{n}{n-m}}\right). \tag{22}$$

Formula (21) is easily obtained from the work of S. M. Nikol' skii ([9]). From these formulas follow (12), (13), and (14).

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Note: Figure translations are in progress. See original paper for figures.

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