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Abstract

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INTERNAL HOMOLOGIES

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1. Statement of the main theorem. In my note ([1]) the natural homomorphism h^k of the group $\mathfrak{S}^k$ of internal homologies into the group $\mathfrak{N}^k$ of internal homologies mod 2 was studied; namely, the kernel of the homomorphism h^k (it is equal to $2\mathfrak{S}^k$) and the image $h^k(\mathfrak{S}^k)$ were found. An exhaustive characterization of this image was, however, impossible until the groups $\mathfrak{N}^k$ themselves had been sufficiently studied. At present their structure is known. Let $[M^k]$ denote the element of the group $\mathfrak{N}^k$ defined by the manifold M^k, and let $\mathfrak{N}$ denote the algebra that is the direct sum of all groups $\mathfrak{N}^k$, with multiplication defined by the usual multiplication of manifolds. As Thom showed ([2]), for each natural number r not of the form $2^s - 1$, there exists a manifold $P(r)$ of dimension r such that the elements $[P(r)]$ form an independent mod 2 system of generators of the algebra $\mathfrak{N}$; in other words, all possible products

$$[P(r_1)] \times [P(r_2)] \times \cdots \times [P(r_l)]$$

with $r_1 + r_2 + \cdots + r_l = k$ form an independent mod 2 system of generators of the group $\mathfrak{N}^k$. Concrete manifolds $P(r)$ for even r and for $r = 5$ were indicated by Thom, and for the remaining r by Dold ([3]). It is precisely these manifolds that we shall denote by $P(r)$. Moreover, for $P(r)$ with odd r we need know only that they are orientable. $P(r)$ with even r is the real projective space $PR(r)$. According to the Pontryagin-Thom theorem, $[M_1^k] = [M_2^k]$ if and only if M_1^k and M_2^k have the same characteristic numbers; a characteristic number of a manifold M^k is a scalar product of the form (ρ, m^k), where ρ is a product of weight k of Stiefel-Whitney classes

$$\left(\rho = \prod_{\alpha=1}^{l} w_{r_\alpha}, \quad r_1 + \cdots + r_l = k\right),$$

and m^k is the fundamental Δ-class mod 2 of the manifold M^k. Relying on these facts, I give in this note an effective description of $h^k(\mathfrak{S}^k)$.

In § 2 it will be shown that

$$PR(n) \times PR(n) \sim PC(n) \quad (\text{mod } 2) \text{ (in)} \tag{1}$$

($PC(n)$ is complex projective space; formula (1) with $n = 2$ is found in ([1,2])). Consequently, the generators

$$[P(r_1)] \times [P(r_2)] \times \cdots \times [P(r_l)],$$

in which each factor of even dimension occurs an even number of times, belong to $h^k(\mathfrak{S}^k)$. They are called **generators of the first kind**; the remaining generators are **generators of the second kind**; and the generators for which all r_α are even and pairwise distinct are **special**. Every generator of the second kind is a product of a generator of the first kind by a special generator. A characteristic number (ρ, m^k) in which ρ contains the class w_1 as a factor is called a w_1**-characteristic number**. The set of manifolds M^k for which ([M^k]

$\in h^k(\mathfrak{S}^k)$, is denoted by K_1; the class of manifolds M^k for which all w_1-numbers are equal to zero, by K_2; the class of manifolds M^k for which $[M^k]$ belongs to the subgroup of the group $\mathfrak{N}^k$ generated by generators of the first kind, by K_3; the class of manifolds M^k for which $A^{k-1} \sim 0 \pmod 2$ (intr.), $B^{k-2} \sim 0 \pmod 2$ (intr.), by K_4 (for the definition of A^{k-1} and B^{k-2} see in (1)); the class of manifolds M^k for which $A^{k-1} \sim 0$ (intr.), $B^{k-2} \sim 0 \pmod 2$ (intr.), by K_5.

Main theorem. $K_1 = K_2 = K_3 = K_4 = K_5$.

It is obvious that $K_1 \subset K_2$, $K_5 \subset K_4$. From formula (1) it follows that $K_3 \subset K_1$. Therefore it is enough to prove formula (1) and the relations $K_2 = K_4$, $K_2 \subset K_3$, $K_1 \subset K_5$. This will be done in §§ 2–5. Some consequences of the main theorem are given in § 6.

2. Proof of formula (1). Let a and b be generators of the one-dimensional groups of V-homology mod 2 of the manifolds $PR(n)$ standing on the left, and let c be a generator of the two-dimensional group of V-homology mod 2 of the manifold $PC(n)$. For the left-hand side

$$w_r = \sum_{i=0}^{r} \binom{n+1}{i} \binom{n+1}{r-i} a^{r-i} \otimes b^i. \tag{2}$$

If w_r enters into the product $\prod_{\alpha=1}^{l} w_{r_\alpha}$ of weight $r_1 + \dots + r_l = 2n$, then the terms of the sum (2) equidistant from the beginning and the end give, upon multiplication by the remaining classes w_{r_α}, identical results. Consequently, when computing the corresponding number they may be discarded, and the class w_r may be replaced by zero if r is odd, and by the middle term

$$\binom{n+1}{s} \binom{n+1}{s} a^s \otimes b^s = \binom{n+1}{s} a^s \otimes b^s,$$

if $r = 2s$. For the right-hand side, $w_r = 0$ if r is odd, and $w_{2s} = \binom{n+1}{s} c^s$. Consequently, the left- and right-hand sides have the same numbers, which proves formula (1).

3. Proof of the relation $K_2 = K_4$. Let m^k, a^{k-1}, b^{k-2} be the fundamental Δ-classes mod 2 of the manifolds M^k, A^{k-1}, B^{k-2}, and let w_r, u_r, v_r be the Stiefel–Whitney classes of these manifolds; let $\bar{u}_r, \bar{v}_r$ be the normal classes determined by the embeddings $i : A^{k-1} \to M^k$, $j : B^{k-2} \to M^k$. The complete skew products with bases A^{k-1}, B^{k-2} determined by these embeddings are sums (in the sense of Whitney) of tangent and normal skew products, and their characteristic classes are i^*w_r and j^*w_r. Hence,

$$i^*w_r = u_r + u_{r-1}\bar{u}_1, \quad j^*w_r = v_r + v_{r-1}\bar{v}_1 + v_{r-2}\bar{v}_2 \quad (r = 1, 2, \dots;\ v_{-1} = 0).$$

From the orientability of A^{k-1} and from the fact that the classes i_*a^{n-1}, j_*b^{n-2} are dual in M^k to the classes w_1, w_1^2, it follows that $u_1 = 0$, $\bar{v}_1 = 0$, $\bar{v}_2 = v_1^2$. Thus, finally:

$$i^*w_r = u_r + u_{r-1}\bar{u}_1 \quad (u_1 = 0); \qquad j^*w_r = v_r + v_{r-2}v_1^2. \tag{3}$$

These equalities can be solved inductively with respect to u_r and v_r:

$$u_r = i^*\varphi_r(w_1, \dots, w_r); \qquad v_r = j^*\psi_r(w_1, \dots, w_r), \tag{4}$$

where φ_r, ψ_r are homogeneous polynomials of weight r. From (4) it follows that for every product $\xi = \prod_{\alpha=1}^{l} u_{r_\alpha}$ of weight $k-1$ there exists such a homogeneous polynomial $\varphi(w_1, \dots, w_{k-1})$ of weight $k-1$, and for every product $\eta = \prod_{\beta=1}^{m} v_{s_\beta}$ of weight $k-2$ such a homogeneous polynomial $\psi(w_1, \dots, w_{k-2})$ of weight

$k-2$, that $\xi = i^*\varphi$, $\eta = j^*\psi$. If $M^k \in \mathrm{K}_2$, then

$$(\xi, a^{k-1}) = (i^*\varphi, a^{k-1}) = (\varphi, i_*a^{k-1}) = (w_1\varphi, m^k) = 0,$$

$$(\eta, b^{k-2}) = (j^*\psi, b^{k-2}) = (\psi, j_*b^{k-2}) = (w_1^2\psi, m^k) = 0.$$

This means that all the residues of A^{k-1} and B^{k-2} are zero, i.e. that $[A^{k-1}] = 0$ and $[B^{k-2}] = 0$, i.e. that $M^k \in \mathrm{K}_4$. Thus, $\mathrm{K}_2 \subset \mathrm{K}_4$.

Now let $M^k \in \mathrm{K}_4$. From (3) it follows that φ_r in formula (4) can be represented in the form $w_r + w_1\omega_{r-1}$, where ω_{r-1} is a polynomial in $w_1, \dots, w_{r-1}$. Consequently, φ in the formula $\xi = i^*\varphi$ can be represented in the form $\rho + w_1\chi$, where

$$\rho = \prod_{\alpha=1}^{l} w_{r_\alpha},$$

and χ is a polynomial in $w_1, \dots, w_{k-2}$. Formula (3) allows $j\chi$ to be represented as some polynomial $\zeta(v_1, \dots, v_{k-2})$. Since $\rho = \varphi + w_1\chi$, we have

$$(w_1\rho, m^k) = (w_1\varphi, m^k) + (w_1^2\chi, m^k) = (\xi, a^{k-1}) + (\eta, b^{k-2}) = 0.$$

This proves that all the w_1-residues of M^k are zero, i.e. that $M^k \in \mathrm{K}_2$. Thus, $\mathrm{K}_4 \subset \mathrm{K}_2$.

4. **Proof of the relation** $\mathrm{K}_2 \subset \mathrm{K}_3$**.** It is enough to prove that if $M^k \in \mathrm{K}_2$ and

$$[M^k] = \sum_{i=1}^{p} [N_i^k],$$

where the $[N_i^k]$ are pairwise distinct generators of the second kind, then $[M^k] = 0$, i.e. $p = 0$. Suppose first that all $[N_i^k]$ are special generators, and let $[N_1^k]$ be the largest among them: if

$$N_1^k = PR(r_1) \times PR(r_2) \times \cdots \times PR(r_l), \quad N_i^k = PR(s_1) \times PR(s_2) \times \cdots \times PR(s_m),$$

with $i > 1$ and $r_1 > r_2 > \cdots > r_l$, $s_1 > s_2 > \cdots > s_m$, $r_\alpha = s_\alpha$ for $\alpha < \beta$, $r_\beta \neq s_\beta$, then $r_\beta > s_\beta$. Define a homomorphism of the ring of symmetric polynomials in $t_1, \dots, t_k$ over the prime field of characteristic 2 into the ring of ∇-homologies mod 2 of a k-dimensional manifold, which assigns to the r-th elementary symmetric function the class w_r; denote by $\sigma(r)$ the class corresponding to the polynomial $t_1^r + \cdots + t_k^r$, and put

$$\tau = \sigma(1)\sigma(r_1 - 1) \times \sigma(r_2) \cdots \sigma(r_l).$$

This is a homogeneous polynomial of weight k in $w_1, \dots, w_k$, and, since $\sigma(1) = w_1$, the corresponding residue is a w_1-residue. Consequently, for M^k it is zero. As the computation shows, for N_1^k and N_i^k with $i > 1$ it is equal, respectively, to 1 and 0, and the relation

$$[M^k] = \sum [N_i^k]$$

gives $0 = 1$.

Now let the $[N_i^k]$ be arbitrary generators of the second kind. Each of them has the form $[O] \times [N]$, where O is a generator of the first kind and N is a special generator. Collecting terms with the same O, we obtain

$$[M^k] = \sum [O_j] \times [M_j],$$

where the $[O_j]$ are pairwise distinct, and the $[M_j]$ are sums of special generators. Construct for M_j the submanifolds A_j and B_j. Analogous submanifolds for $\sum O_j \times M_j$ may be taken to be $\sum O_j \times A_j$ and $\sum O_j \times B_j$, and, since $\sum O_j \times M_j$, together with M^k, belongs to $\mathrm{K}_2 = \mathrm{K}_4$, it follows that

$$\left[\sum O_j \times A_j\right] = 0, \qquad \left[\sum O_j \times B_j\right] = 0.$$

But this is possible only under the condition that $[A_j] = 0,\ [B_j] = 0$, i.e. only under the condition that $M_j \in \mathrm{K}_4 = \mathrm{K}_2$. According to what was proved in the preceding paragraph, it follows that $[M_j] = 0$; thus $[M^k] = 0$.

5. **Proof of the relation** $\mathrm{K}_1 \subset \mathrm{K}_5$**.** Let $M^k \subset \mathrm{K}_1$, and let L^{k+1} be a manifold with boundary $M^k + M_1^k$, where M_1^k is an orientable manifold. Construct in L^{k+1} an orientable submanifold A^k, serving as

which is a k-dimensional (integer) Stiefel Δ-cycle. Next, in A^k we construct a submanifold B^{k-1} serving as a Δ-cycle $(\mathrm{mod} 2)$ of the singularities of the external vector field on A^k in L^{k+1}. Owing to orientability, the submanifolds A^k and B^{k-1} of M_1^k can be chosen so that the oriented boundary for A^k is A^{k-1}, and the boundary for B^{k-1} is B^{k-2}. Therefore $A^{k-1} \sim 0$ (int.) and $B^{k-2} \sim 0 \bmod 2$ (int.), i.e. $M^k \in K_5$.

6. Consequences of the main theorem.

A. *In the algebra* $\mathfrak{N}$*, the product of a nonorientable element by a nonzero orientable element is a nonorientable element.*

This follows, for example, from the equality $K_1 = K_3$.

B. *If* $M^k \sim 0 \bmod 2$ *(int.) and* $2M^k \sim 0$ *(int.), then* $M^k \sim 0$ *(int.).*

Proof. Let L^{k+1} be an oriented manifold with boundary $2M^k$, and let M^{k+1} be a nonorientable closed manifold into which the manifold L^{k+1} is transformed under the natural identification of two copies of the manifold M^k constituting its boundary. M^k serves as a k-dimensional Stiefel Δ-cycle of the manifold M^{k+1} and possesses in M^{k+1} an external vector field without singularities. Consequently, $M^{k+1} \in K_4$, and, since $K_4 = K_5$, it follows that $M^k \sim 0$ (int.).

C. *In* $\mathfrak{N}^k$ *there are no elements of finite order divisible by 4.*

Proof. Suppose $4mM^k \sim 0$ (int.). Put $M_1^k = 2mM^k$. Then $M_1^k \sim 0 \bmod 2$ (int.), $2M_1^k \sim 0$ (int.). Consequently, $M_1^k \sim 0$ (int.), and $4m$ is not the order of the manifold M^k.

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Note: Figure translations are in progress. See original paper for figures.