MATHEMATICS

M. I. KADEC

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Abstract

Full Text

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M. I. KADEC

ON WEAK AND STRONG CONVERGENCE

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The following proposition is known, connecting the notions of weak and strong convergence:

Theorem 1. *If a sequence x_n of elements of a uniformly convex space converges weakly to an element x and $\lim_{n\to\infty} \|x_n\| = \|x\|$, then the sequence x_n converges to x strongly* ([1]).

This theorem was extended by Vyborny ([2]) to locally uniformly convex spaces ([3]).

In the present note the following will be proved:

Theorem 2. *In any separable Banach space one can introduce a new norm, equivalent to the old one, such that for arbitrary elements x_n and x from*

$$x_n \xrightarrow{\text{sl}} x, \qquad \lim_{n\to\infty} \|x_n\| = \|x\| \tag{1}$$

it follows that

$$\lim_{n\to\infty} \|x_n - x\| = 0. \tag{2}$$

We shall prove this theorem for the space C of continuous functions defined on $[0,1]$, and then use the universality of C in the class of separable Banach spaces.

On the set of continuous functions on $[0,1]$, define the norm

$$\|f(t)\| = \max_{0\le t\le 1} |f(t)| + \sum_{k=1}^{\infty} \frac{1}{2^k}\,\omega\!\left(f, \frac{1}{k}\right), \tag{3}$$

where $\omega(f,\delta)$ is the modulus of continuity of the function $f(t)$:

$$\omega(f,\delta) = \max_{|t'-t''|\le\delta} |f(t') - f(t'')|.$$

Since $\omega(f,\delta) \leq 2 \max\limits_{0\leq t\leq 1} |f|$, the norm (3) is equivalent to the norm of the space C:

$$\max_{0\leq t\leq 1} |f(t)| \leq \|f(t)\| \leq 3 \max_{0\leq t\leq 1} |f(t)|.$$

The space of continuous functions with the norm (3) will be denoted by C^*. Let us prove some auxiliary propositions.

Lemma 1. *If a sequence $\varphi_n(t)$ converges to zero at every point of the interval $[0,1]$ and $\|\varphi_n\| = 1$, then for any $\delta > 0$*

$$\lim_{n\to\infty} \omega(\varphi_n, \delta) \geq \frac{1}{3}. \tag{4}$$

Take n_0 so large that for all $n > n_0$ and arbitrary $\varepsilon > 0$ the inequality $|\varphi_n(t)| < \varepsilon$ holds on a set E, whose measure ⋯

greater than $1-\delta$. If $\varphi_n(t_n') = \max_{0\leq t\leq 1} |\varphi_n(t)|$, $t_n'' \in E$, and $|t_n' - t_n''| < \delta$, then

$$|\varphi_n(t_n') - \varphi_n(t_n'')| > \max_{0\leq t\leq 1} |\varphi_n(t)| - \varepsilon,$$

whence (4) follows.

Lemma 2. If the sequence $\varphi_n(t)$ converges to zero at every point of the interval $[0,1]$, then

$$\lim_{n\to\infty} \omega(f+\varphi_n, \delta) \geq \omega(f,\delta) \tag{5}$$

for every continuous function $f(t)$ and $\delta > 0$.

Let $|t' - t''| \leq \delta$ and $|f(t') - f(t'')| = \omega(f,\delta)$. Choose n_0 so large that, for all $n > n_0$ and arbitrary $\varepsilon > 0$, the inequalities $|\varphi_n(t')| < \varepsilon/2$ and $|\varphi_n(t'')| < \varepsilon/2$ hold; then

$$|f(t') + \varphi_n(t') - f(t'') - \varphi_n(t'')| \geq |f(t') - f(t'')| - \varepsilon,$$

whence (5) follows.

Now let $\varphi_n \in C^*$ be a sequence weakly converging to zero such that $\|\varphi_n\| > 3\varepsilon > 0$, and let f be an arbitrary element of C^*. Define the index $q = q(\varepsilon)$ so that

$$\omega\left(f, \frac{1}{k}\right) < \frac{\varepsilon}{8} \quad \text{for } k > q. \tag{6}$$

Choose n so as to satisfy the conditions

$$\sum_{k=q+1}^{\infty} \frac{1}{2^k} \omega\left(\varphi_n, \frac{1}{k}\right) > \frac{\varepsilon}{2^{q+1}}, \tag{7}$$

which can be done on the basis of (4), and

$$\omega\left(f+\varphi_n, \frac{1}{k}\right) > \omega\left(f, \frac{1}{k}\right) - \frac{\varepsilon}{2^{q+3}} \quad \text{for } k \leq q. \tag{8}$$

Split the sum in (3) into two:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \omega\left(f+\varphi_n, \frac{1}{k}\right) = \sum_{k=1}^{q} \frac{1}{2^k} \omega\left(f+\varphi_n, \frac{1}{k}\right) + \sum_{k=q+1}^{\infty} \frac{1}{2^k} \omega\left(f+\varphi_n, \frac{1}{k}\right).$$

Estimate from below each of the sums obtained:

$$\sum_{k=1}^{q} \frac{1}{2^k} \omega\left(f+\varphi_n, \frac{1}{k}\right) > \sum_{k=1}^{q} \frac{1}{2^k} \left[\omega\left(f, \frac{1}{k}\right) - \frac{\varepsilon}{2^{q+3}}\right] > \sum_{k=1}^{q} \frac{1}{2^k} \omega\left(f, \frac{1}{k}\right) - \frac{\varepsilon}{2^{q+3}} >$$

$$> \sum_{k=1}^{\infty} \frac{1}{2^k} \omega\left(f, \frac{1}{k}\right) - \sum_{k=q+1}^{\infty} \frac{1}{2^k} \frac{\varepsilon}{8} - \frac{\varepsilon}{2^{q+3}} = \sum_{k=1}^{\infty} \frac{1}{2^k} \omega\left(f, \frac{1}{k}\right) - \frac{\varepsilon}{2^{q+2}}; \tag{9}$$

in this estimate we have used inequalities (8) and (6).

We estimate the second sum with the aid of inequalities (7) and (6):

$$\sum_{k=q+1}^{\infty} \frac{1}{2^k} \omega\left(f+\varphi_n, \frac{1}{k}\right) \geq \sum_{k=q+1}^{\infty} \frac{1}{2^k} \omega\left(\varphi_n, \frac{1}{k}\right) - \sum_{k=q+1}^{\infty} \frac{1}{2^k} \omega\left(f, \frac{1}{k}\right) >$$

$$> \frac{\varepsilon}{2^{q+1}} - \frac{\varepsilon}{2^{q+3}} = \frac{3\varepsilon}{2^{q+3}}. \tag{10}$$

Adding (9) and (10), we obtain

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \omega\left(f+\varphi_n, \frac{1}{k}\right) > \sum_{k=1}^{\infty} \frac{1}{2^k} \omega\left(f, \frac{1}{k}\right) + \frac{\varepsilon}{2^{q+3}}.$$

Since, moreover,

$$\lim_{n\to\infty} \max_{0\le t\le 1} |f+\varphi_n| \ge \max_{0\le t\le 1} f,$$

we have

$$\lim_{n\to\infty} \|f+\varphi_n\| > \|f\|,$$

whence the validity of Theorem 2 follows.

In what follows we shall consider spaces satisfying conditions (1), (2) and, in addition, strictly normed. Such a norm can be obtained by modifying (3) in the following way:

$$\|f\| = \max_{0\le t\le 1} |f(t)| + \left[\int_0^1 f^2(t)\,dt\right]^{1/2} + \sum_{k=1}^{\infty} \frac{1}{2^k}\,\omega\left(f, \frac{1}{k}\right). \tag{3a}$$

Let us apply Theorem 2 to the proof of the following assertion.

Theorem 3. *Separable reflexive Banach spaces are monotonically equivalent.*

Let X be a separable reflexive space satisfying (1), (2) and strictly normed. Consider a system of linear subspaces $P_0 \supset P_1 \supset P_2 \supset \cdots$ such that $\operatorname{def} P_n = n$, and the intersection of all P_n contains only the zero element of the space; such a system exists in every separable Banach space. For each element $x \in X$ introduce the sequence of deviations $H_n(x)$:

$$H_n(x) = \min_{y\in P_n} \|x-y\| = \|x - x^{(n)}\| \qquad (n = 1, 2, \ldots).$$

The existence and uniqueness of the element of best approximation $x^{(n)}$ are ensured respectively by the reflexivity and the strict normedness of the space X. For every n, evidently,

$$H_n(x) \le H_{n+1}(x) \le \|x\|.$$

We shall use the fact that each P_n divides P_{n-1} into three parts

$$P_{n-1} = P_{n-1}^{+} + P_n + P_{n-1}^{-},$$

and construct a sequence of functionals $\varepsilon_n(x)$:

$$\begin{aligned}
\varepsilon_n(x) &= +1, && \text{if } x^{(n-1)} \in P_{n-1}^{+};\\
\varepsilon_n(x) &= \varepsilon_{n-1}(x), && \text{if } x^{(n-1)} \in P_n \quad (\text{i.e. } H_{n-1}(x) = H_n(x));\\
\varepsilon_n(x) &= -1, && \text{if } x^{(n-1)} \in P_{n-1}^{-}.
\end{aligned}$$

In addition, we shall put $\varepsilon_0(x) = 0$ for every x.

Lemma 3. *Let $h_1, h_2, \dots, h_n$ be a sequence of real numbers satisfying the conditions*

$$|h_j| \leq |h_{j+1}|; \qquad \text{if } |h_j| = |h_{j+1}|, \text{ then } h_j = h_{j+1}. \tag{11}$$

The set Q_n of elements x such that

$$\varepsilon_k(x) H_k(x) = h_k \quad (k \leq n), \tag{12}$$

is nonempty and is obtained from P_n if to each $y \in P_n$ one adds one and the same element x^*, satisfying (12).

The proof of this lemma is given in ([4]).

Lemma 4. *Whatever bounded sequence of real numbers $h_1, h_2, h_3, \dots$ satisfying (11) may be, there exists a unique element x for which*

$$\varepsilon_n(x) H_n(x) = h_n \quad (n = 1, 2, \dots); \tag{13}$$

moreover $\|x\| = \lim_{n\to\infty} H_n(x)$.

The set of elements satisfying (13) is the intersection $\prod_1^\infty Q^n$ of the sets satisfying (12). It is therefore sufficient to show,

that this intersection contains exactly one point. Suppose that

$$\prod_1^\infty Q_n$$

contains the element x; then, according to Lemma 4, each $Q_n = x + P_n$, and the whole family $\{Q_n\}_1^\infty$ is congruent to $\{P_n\}_1^\infty$. Since

$$\prod_1^\infty P_n$$

by definition contains a single element, it follows that

$$\prod_1^\infty Q_n$$

also contains a single element. It remains to show that

$$\prod_1^\infty Q_n$$

is nonempty. Consider the intersections of the sets Q_n and of the ball S with arbitrary radius

$$h > \lim_{n\to\infty} |h_n|.$$

The resulting nonempty closed convex sets form a decreasing sequence. By a theorem of V. L. Shmul' yan [5], the intersection of these sets is nonempty. Thus, the existence and uniqueness of the element x are proved. Since h may be taken arbitrarily close to $\lim_{n\to\infty} |h_n|$, the equality

$$\|x\| = \lim_{n\to\infty} H_n(x)$$

has thereby also been proved.

Before passing to the proof of Theorem 3, let us note that if the sequence x_n converges weakly to x, then the element x_n, as n increases, approaches without bound each Q_k containing x, and therefore

$$\begin{aligned} H_k(x) &= \lim_{n\to\infty} H_n(x_n) \quad (k = 1, 2, ...); \\ \varepsilon_k(x) &= \lim_{n\to\infty} \varepsilon_k(x_n), \quad \text{if } H_{k-1}(x) < H_k(x); \\ \varepsilon_k(x) &= \varepsilon_{k-1}(x), \quad \text{if } H_{k-1}(x) = H_k(x). \end{aligned} \tag{14}$$

Now let X and Y be separable reflexive spaces satisfying (1), (2) and strictly normed; suppose that for the elements of each of them the deviations $H_k(x)$ and $H_k(y)$ and the functionals $\varepsilon_k(x)$ and $\varepsilon_k(y)$ are defined. To each element $x \in X$ assign an element $y \in Y$ so that

$$\varepsilon_k(x) H_k(x) = \varepsilon_k(y) H_k(y) \quad (k = 1, 2, ...). \tag{16}$$

This correspondence is one-to-one. We shall show that it is continuous. Let x_n and x be arbitrary elements of X, and

$$\lim_{n\to\infty} x_n = x. \tag{16}$$

Since Y is reflexive, the corresponding sequence y_n is weakly compact. According to (14), no subsequence of it can converge weakly to an element different from the element y corresponding to x, and therefore

$$y_n \xrightarrow{\text{w}} y. \tag{17}$$

According to Lemma 4, $\|x_n\| = \|y_n\|$, $\|x\| = \|y\|$, whence

$$\lim_{n\to\infty} \|y_n\| = \|y\|. \tag{18}$$

From (17) and (18) it follows that

$$\lim_{n\to\infty} y_n = y.$$

The continuity of the inverse correspondence is proved in exactly the same way. Thus, the spaces X and Y are homeomorphic, which proves Theorem 3.

Kharkov Automobile and Highway Institute

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Note: Figure translations are in progress. See original paper for figures.