HYDROMECHANICS

1958

SovietRxiv

Abstract

Full Text

HYDROMECHANICS

V. P. KOROBEINIKOV

ONE-DIMENSIONAL SELF-SIMILAR MOTIONS OF A CONDUCTING GAS IN A MAGNETIC FIELD

(Presented by Academician L. I. Sedov on 1 IV 1958)

1. Let us consider one-dimensional nonstationary adiabatic motions of a perfectly electrically conducting gas with cylindrical and plane waves. The magnetic field **H** is directed perpendicular to the trajectories of motion of the particles (in the cylindrical case, along the axis of symmetry or tangent to concentric circles centered on this axis). The conductivity of the gas is regarded as infinite; viscosity and thermal conductivity are neglected. Under the assumptions made, the equations of motion have the form

$$\frac{dv}{dt} = -\frac{1}{\rho}\left[\frac{\partial}{\partial r}(p+h) + \frac{2h}{r}(1-n)\right], \qquad \frac{d\rho}{dt} = -\rho\left[\frac{\partial v}{\partial r} + (\nu-1)\frac{v}{r}\right],$$

$$\frac{dp}{dt} = -\gamma p\left[\frac{\partial v}{\partial r} + (\nu-1)\frac{v}{r}\right], \qquad \frac{dh}{dt} = -2h\left[\frac{\partial v}{\partial r} + (\nu-1)\frac{nv}{r}\right], \tag{1}$$

where

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}$$

and so on; $h = \dfrac{H^2}{8\pi}$; H is the magnetic-field intensity; $\nu = 2, 1$; $n = 0$ for a cylindrical magnetic field, $n = 1$ for a field directed along the axis of symmetry ($\nu = 2$); the remaining notation is standard. Since equations (1) contain no dimensional constants, the motion will be self-similar ([1]) if only two dimensional constants with independent dimensions enter the initial and boundary conditions of the problem.

The general theory of unsteady self-similar motions of a liquid and gas in the absence of a magnetic field was developed by L. I. Sedov ([1]). We shall use L. I. Sedov' s notation and terminology.

Suppose that among the determining parameters there are two constants a and b with independent dimensions, where $[a] = ML^kT^s,\ [b] = LT^{-\delta}$.

Introduce the dimensionless variables

$$v = \frac{r}{t}V, \qquad \rho = \frac{a}{r^{k+3}t^{s}}R, \qquad p = \frac{a}{r^{k+1}t^{s+2}}\mathcal{P}, \qquad h = \frac{a}{r^{k+1}t^{s+2}}\mathcal{H}, \qquad \lambda = \frac{r}{bt^{\delta}}.$$

By virtue of self-similarity, the dimensionless functions $V, R, \mathcal{P}, \mathcal{H}$ depend only on a single dimensionless variable λ. The system of partial differential equations (1) is equivalent to the following system of ordinary differential equations:

$$\lambda\left[(V-\delta)V' + \frac{(\mathcal{H}+\mathcal{P})'}{R}\right] = -V^2 + V + (k+1)\frac{\mathcal{H}+\mathcal{P}}{R} - \frac{2(1-n)\mathcal{H}}{R}; \quad (2)$$

$$\lambda\left[(V-\delta)\frac{R'}{R} + V'\right] = s + (k-\nu+3)V; \quad (3)$$

$$\lambda\left[(V-\delta)\frac{\mathcal{P}'}{\mathcal{P}} + \gamma V'\right] = s + 2 + (k+1-\gamma\nu)V; \quad (4)$$

$$\lambda\left[(V-\delta)\frac{\mathcal{H}'}{2\mathcal{H}} + V'\right] = \frac{s+2}{2} + \left[\frac{k-1}{2} + (1-\nu)n\right]V. \quad (5)$$

If the constants $\gamma,\ \delta$ are related to $s,\ k,\ \nu,\ n$ by the relations

$$2(\gamma\nu - k) = (2-\nu+\gamma\nu)(\delta+s+2), \quad (2-\gamma\nu-2(1-\nu)n = 0,$$

then the system (2)–(5) has the particular solution

$$V = A, \qquad \mathcal{P} = B\lambda, \qquad \rho = C\lambda, \qquad \mathcal{H} = D\lambda,$$

where B and D are arbitrary positive constants;

$$A = \frac{2}{2-\nu+\gamma\nu}, \qquad C = \frac{k(B+D) - 2(1-n)D}{A(A-1)}.$$

For this solution the dimensional velocity depends linearly on the coordinate r. More general solutions of this type are considered in the papers ([2],[3]).

Using the conservation laws and methods of dimensional theory ([1],[4]), one can show that equations (3)–(5) have two algebraic integrals:

1) The adiabaticity integral ([1],[5])

$$\frac{\mathcal{P}}{R^{\gamma}} = [R(V-\delta)]^{\frac{2-(\gamma-1)s+\delta[k+1-\gamma(k+3)]}{\mu}} \lambda^{-\frac{[2+\nu(\gamma-1)]s+2(k+3-\nu)}{\mu}} x_1, \tag{6}$$

2) The frozen-in integral

$$\frac{V\mathcal{H}}{\lambda^{1-n}R} = [R(V-\delta)]^{\frac{2-s-\delta(k+2n+3)}{2\mu}} \lambda^{-\frac{(k+5)\mu-(\nu-k-3)[2-s-\delta(k+2n+3)]}{2\mu}} x_2, \tag{7}$$

where x_1 and x_2 are arbitrary constants; $\mu = s + \delta(k+3-\nu)$.

Thus the solution of all self-similar problems is reduced to the integration of a system of two ordinary equations.

If

$$s + 2 - \delta(\gamma - 1 - k) = 0,$$

then for the system (2)—(5) there exists an energy integral

$$\lambda^{\nu+2}\left[(\mathcal{P}+\mathcal{H})V - (V-\delta)\left(\frac{RV^2}{2} + \frac{\mathcal{P}}{\gamma-1} + \mathcal{H}\right)\right] = \text{const}. \tag{8}$$

In this case the problem is reduced to the solution of a single equation.

During the motion of the gas, shock waves may arise. The conditions on shock waves, which are consequences of the conservation laws, for the self-similar motions under consideration may be written as follows:

$$\{R(V-\delta)\} = 0, \qquad \{\mathcal{H}(V-\delta)^2\} = 0, \qquad \{R(V-\delta)V + \mathcal{P} + \mathcal{H}\} = 0,$$

$$\left\{R(V-\delta)\left(\frac{\mathcal{P}}{R(\gamma-1)} + \frac{V^2}{2} + \frac{\mathcal{H}}{R}\right) + (\mathcal{P}+\mathcal{H})V\right\} = 0. \tag{9}$$

Here it has been taken into account that for the velocity c of the shock wave there is the dependence

$$c = \delta r_2/t,$$

where r_2 is the radius of the shock wave; braces denote the difference of the values of the quantities on the two sides of the discontinuity surface. For flows with shock waves, (9) are boundary conditions in finding the functions $V(\lambda)$, $R(\lambda)$, $\mathcal{P}(\lambda)$, $\mathcal{H}(\lambda)$.

Fig. 1

Figure 1: Fig. 1

2. The following self-similar problems may be indicated, the solution of which is reduced to the integration of the system (2)–(5).

1) The problem of the motion of a conducting gas from prescribed initial data (Cauchy problem). From the requirement of self-similarity it follows that the initial distributions (at $t = 0$) will have the form:

$$v_0 = \alpha_1 b^{\frac{1}{\delta}} r^{1-\frac{1}{\delta}}, \qquad \rho_0 = \alpha_2 a b^{\frac{s}{\delta}} r^{-(k+3+\frac{s}{\delta})},$$

$$p_0 = \alpha_3 a b^{\frac{s+2}{\delta}} r^{-(k+1+\frac{s+2}{\delta})}, \qquad h_0 = \alpha_4 p_0,$$

where α_i $(i = 1, \dots, 4)$ are prescribed dimensionless constants. In the plane case the initial data may have a discontinuity at $r = 0$.

The simplest example of this problem is the problem of the decay of an arbitrary discontinuity, when at $t = 0$ the constant quantities

$$v = v_1, \qquad p = p_1, \qquad \rho = \rho_1, \qquad h = h_1 \quad \text{for } r < 0;$$

$$v = v_2, \qquad p = p_2, \qquad \rho = \rho_2, \qquad h = h_2 \quad \text{for } r > 0;$$

the velocities v_1 and v_2 are directed toward one another; their differences are large in absolute value. As a result of the decay of an arbitrary discontinuity, shock waves propagate in both directions; they move with constant speed and have constant intensity.

Using conditions (9), one can calculate all the characteristics of the resulting motion. Motion of this kind may arise, for example, in the collision of cosmic gas masses. A detailed solution of this problem for $h = 0$ is given in (7).

Fig. 1

2) The problem of the motion of a plane or cylindrical (conducting) piston in a gas. At the initial moment of time $v_1 = 0$, p_1, ρ_1, and h_1 are constant, and the piston begins to move with constant speed U. The problem is self-similar.

Let us consider the solution for a plane piston. Ahead of the piston a shock wave propagates through the gas with constant speed c. In the region of motion behind the shock front, $v = v_2 = U$; $p = p_2$, $\rho = \rho_2$, $h = h_2$ are constant quantities.

Using the conditions at the shock wave (9), one can find the dependence of c, p, ρ, h on the piston velocity U and on ρ_1, p_1, h_1. Figure 1 shows the dependence of c/a_{*1} on U/a_{*1} for $\gamma = 1.4$ and $h_1/p_1 = 0$, $h_1/p_1 = 1$, where

$$a_{*1}^2 = \frac{\gamma p_1}{\rho_1} + \frac{2h_1}{\rho_1}.$$

3) The problem of a strong explosion (electric discharge). At the moment $t = 0$ in a gas at rest, an instantaneous release of finite energy E_0 occurs along a straight line, i.e., an explosion occurs. E_0 is calculated per unit length. This explosion may be regarded as a high-intensity electric discharge in a gas, occurring along a straight line. The initial density and magnetic-field intensity are variable:

$$\rho_1 = A_1 r^{-\omega}, \qquad h_1 = B_1 r^{-2}, \qquad \omega < 3.$$

For a strong explosion, the influence of the initial pressure p_1 may be neglected (1). Note that, for any constant p_1, the indicated value of h_1 in a cylindrical field satisfies the equilibrium equation

$$\frac{\partial}{\partial r}(p + h) + \frac{2h}{r} = 0.$$

In this problem there exists an energy integral (8), i.e., its solution reduces to the integration of one ordinary first-order differential equation.

By analogy with (1), formulations of self-similar problems of detonation and combustion in a gas in the presence of a magnetic field are also possible. For $\nu = 2$ one can formulate self-similar problems with magnetic lines of force having the form of helical lines.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
28 III 1958

REFERENCES

1. L. I. Sedov, *Similarity and Dimensional Methods in Mechanics*, 4th ed., Moscow, 1957.
2. A. G. Kulikovskii, DAN, **115**, No. 4 (1957).
3. I. M. Yavorskaya, DAN, **115**, No. 4 (1957).

4. V. P. Korobeinikov, DAN, **104**, No. 4 (1955).

5. M. L. Lidov, DAN, **103**, No. 1 (1955).

6. F. Hoffmann, E. Teller, Phys. Rev., **80**, 692 (1950).

7. L. D. Landau, E. M. Lifshitz, *Mechanics of Continuous Media*, 2nd ed., Moscow, 1954.

[[unclear: the page image contains no visible text]]

Note: Figure translations are in progress. See original paper for figures.