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Abstract

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MATHEMATICS

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ON THE DIFFERENTIAL PROPERTIES OF GENERALIZED SOLUTIONS OF CERTAIN MULTIDIMENSIONAL VARIATIONAL PROBLEMS

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Let us consider the problem of finding the infimum of the functional

$$I(u) = \int_\Omega F(u_1, \dots, u_n)\, dx, \qquad \text{where } dx = dx_1 \dots dx_n, \quad u_i = \frac{\partial u}{\partial x_i}.$$

Put

$$|u(x)|_1 = \left[\sum_{k=1}^n u_k^2(x)\right]^{1/2}, \qquad F_i = \frac{\partial F}{\partial u_i}, \qquad F_{ij} = \frac{\partial^2 F}{\partial u_i \partial u_j}.$$

We shall assume: Ω is a bounded domain of n-dimensional Euclidean space; F is a twice continuously differentiable function of its arguments, satisfying the conditions

$$m_1 |u|_1^p \leqslant F(u_k) \leqslant M_1(|u|_1^p + 1), \qquad m_2 > 0, \qquad p \geqslant 2; \tag{1}$$

$$F_{ij}(u_k) \geqslant m_2(|u|_1^{p-2} + 1), \qquad m_2 > 0; \tag{2}$$

$$|F_i(u_k)| \leqslant M_2(|u|_1^{p-1} + 1); \qquad |F_{ij}(u_k)| \leqslant M_3(|u|_1^{p-2} + 1). \tag{3}$$

The boundary values of the functions admissible for comparison are determined by a function $\varphi(x)$, which we regard as given on all of Ω and as belonging to the space $W_p^1(\Omega)$. The space $W_p^k(\Omega)$ consists of all functions summable over Ω

to the power p, together with their generalized derivatives up to order k. Its norm is defined by the equality

$$\|u\|_{W_p^k} = \left\{ \int_\Omega \left[|u|^p + \sum_{|s|=1}^{k} \sum_{(s)} |D^{(s)} u|^p \right] dx \right\}^{1/p}.$$

In the work [1], simple necessary and sufficient conditions were obtained for the possibility of such an extension of φ from the boundary to the whole domain Ω.

For the functional I we study the following variational problem:

Find $\inf I(u)$ among all functions u satisfying the condition

$$u(x) - \varphi(x) \in \overset{\circ}{W}{}_p^1(\Omega).$$

A function attaining this infimum will be called a generalized solution of the problem.

The space $\overset{\circ}{W}{}_p^1(\Omega)$ is obtained by closure, in the norm $W_p^1(\Omega)$, of the set of all functions continuously differentiable in Ω and equal to zero near the boundary.

The unique solvability of this variational problem follows elegantly and easily from conditions (1) and (2) and the weak compactness of the unit sphere

of the space W_p^1 (this was first established by Morrey [2]; for the most recent works see the paper of V. I. Kazimirov [3]). We prove that, under the conditions listed, the following holds:

Theorem 1. The generalized solution u of the variational problem has, inside Ω, generalized second derivatives u_{ij} and satisfies almost everywhere the Euler equation $F_i u_{ij} = 0$. For it the integral

$$\int_{\Omega_1} (|u_i|^{p-2} + 1) \sum_{i,j} u_{ij}^2 \, dx \leqslant \text{const} \tag{4}$$

is finite for any strictly interior subdomain Ω_1 of the domain Ω. If $\varphi(x) \in W_p^2(\Omega)$ and the boundary S of the domain Ω is twice boundedly differentiable, then the integral (4) is bounded for $\Omega_1 = \Omega$.

Suppose further that $\varphi(x)$ is continuously differentiable and that the boundary $(n-1)$-dimensional surface $F\{x \in S,\ u|_S = \varphi|_S\}$ in the $(n+1)$-dimensional space (x, u) has the following property:

(Δ). If one takes any point $M(x_0, u_0)$ on F and passes through it the tangent $(n-1)$-plane Π_{n-1} to F, and then passes through Π_{n-1} two hyperplanes Π_n and Π_n', making with the hyperplane $u = u_0$ the angles $\pm(\pi/2 - \varepsilon)$, $\varepsilon > 0$, then

F must lie in that dihedral angle formed by Π_n and Π'_n which does not contain the straight line $x = x_0$.

This property (Δ) is a generalization of the well-known "three-point property" to the n-dimensional case.

Theorem 2. If, in addition to the conditions of Theorem 1, it is known that φ has continuous first derivatives and property (Δ) is fulfilled, then the generalized solution u of the variational problem is a continuous function satisfying a Lipschitz condition.

We shall outline the main course of the proof of these theorems and, for brevity of exposition, shall assume that $p = 2$. The first part of Theorem 1 is established as follows: for the generalized solution u the identity

$$\delta I(u) = \int_\Omega F_i(u_k)\,\eta_i(x)\,dx = 0 \tag{5}$$

holds for every $\eta \in \overset{\circ}{W}{}^1_2$. Take in it an $\eta(x)$ equal to zero in some boundary strip, so that $\eta(x - \Delta x_l)$ also belongs to $\overset{\circ}{W}{}^1_2$, and (5) is true for it. Then

$$0 = \frac{1}{\Delta x_l}\left\{\int_\Omega F_i(u_k(x))\,\eta_i(x - \Delta x_l)\,dx - \int_\Omega F_i(u_k(x))\,\eta_i(x)\,dx\right\} =$$

$$= \int_\Omega a_{ij}(x, \Delta x_l)\,u_{(l)j}(x)\,\eta_i(x)\,dx, \tag{6}$$

where it is put

$$u_{(l)}(x) = \frac{u(x + \Delta x_l) - u(x)}{\Delta x_l},$$

$$a_{ij}(x, \Delta x_l) = \int_0^1 F_{ij}(\tau u_k(x + \Delta x_l) + (1 - \tau)u_k(x))\,d\tau.$$

Take in (6) as $\eta(x)$ the function $u_{(l)}(x)\zeta(x)$, where $\zeta(x)$ is a continuously differentiable nonnegative function, equal to zero in a boundary strip, equal to one in $\Omega_1 \subset \Omega$, and satisfying in Ω the condition $|\zeta_i|^2 \le c_1|\zeta|$. Then from (6) it will follow that

$$\int_\Omega a_{ij}u_{(l)j}u_{(l)i}\zeta\,dx = -\int_\Omega a_{ij}u_{(l)j}u_{(l)}\zeta_i\,dx,$$

whence it is not hard to conclude that

$$\int\limits_{\Omega} \zeta \sum_{l=n}^{n} u_{(l)l}^2 \, dx \le c_2 \int\limits_{\Omega} u_{(l)}^2 \, dx \le c_3, \tag{7}$$

where c_3 does not depend on Δx_l. Hence, in turn, it follows that u has generalized second derivatives in Ω, square-summable over any interior subdomain, and (by virtue of (5)) satisfies the Euler equation.

Now let $\varphi \in W_2^2$, and let the boundary S be twice boundedly differentiable. We introduce new coordinates (y) so that some piece of the boundary S_1 lies on the plane $y_n = 0$. As functions $\eta(y)$ we take one of the functions

$$[u_{(l)}(y) - \varphi_{(l)}(y)]\zeta(y), \qquad l \le n-1,$$

where $\zeta(y)$ is a function of the same type as above, but this time it is different from zero near S_1. For such an η, identity (6) is also valid for $l \le n-1$, and therefore so is inequality (7). Hence we conclude that all derivatives $u_{li}(y)$, $l \le n-1$, $i \le n$, are square-summable in some neighborhood of S_1, and, by virtue of the Euler equation, the same will also be true for the derivative $u_{nn}(y)$. In a finite number of steps we thus establish the square summability of u_{li} throughout all of Ω. After this, from (6) we obtain the conclusion that the identity

$$\int\limits_{\Omega} F_{ij}(u_k) u_{lj} \eta_l \, dx = 0 \tag{8}$$

holds for all $\eta \in \overset{\circ}{W}{}_2^1$ (and not only for $\eta \in \overset{\circ}{W}{}_2^1$ that are equal to zero in the boundary strip).

We proceed to the proof of the second theorem. From the conditions on the boundary surface F and from the fact that no caps can be cut off from the surface $u = u(x)$ by any plane, it follows that $\max\limits_S |u|_1 \le c_4$ (we note that it makes sense to speak of u_i on S, since $u \in W_2^2(\Omega)$). We further prove that, by virtue of $\max\limits_S |u|_1 \le c_4$ and $u \in W_2^2(\Omega)$, the functions

$$\xi = \begin{cases} u_l(x) - N, & u_l > N, \\ 0, & u_l < N, \end{cases}$$

will belong to $\overset{\circ}{W}{}_2^1(\Omega)$ for $N > mc_4$, where m is a certain fixed constant determined only by the boundary S. Therefore in (8) one may take $\eta = \xi$. This gives

$$\int\limits_{\{u_l \geq N\}} F_{ij} u_{lj} u_{li}\, dx = 0,$$

whence, by virtue of (2), it follows that almost everywhere in Ω, $u_l \leq N$. Similarly we prove $u_N \geq -N$. From the boundedness of the generalized first derivatives it follows that u is equivalent to a continuous function satisfying the Lipschitz condition. Theorem 2 is proved.

The following generalizations of Theorems 1 and 2 are proved in a completely analogous way.

Theorem $1'$**.** *The assertions of Theorem 1 are valid for*

$$I(u) = \int\limits_{\Omega} F(x, u, u_k)\, dx, \tag{9}$$

if F, in addition to conditions (1)–(3), also satisfies the following:

$$|F_{in}| \leq M\left(|u|_1^{p-2} + 1\right), \qquad |F_{ix_k}| \leq M\left(|u|_1^{p-1} + 1\right).$$

Theorem 2′. The assertion of Theorem 2 is valid for (9), if F satisfies the conditions of Theorems $1'$ and 2 and, for $|u_1| \geq N$, has the form $F(u_k)$.

Many works have been devoted to the study of the differential properties of solutions of variational problems; however, the greater part of them concern the case of two spatial variables (S. N. Bernstein, Tonelli, Morrey, A. G. Sigalov, Cesari, Nirenberg, and others; for recent works see [5]). For $n > 2$ considerably fewer results have been obtained ([4,6] and others). Judging from the abstracting journals, Italian mathematicians have worked extensively on these questions (Miranda, Stampacchia [8], Giorgi, and others). The continuity of a generalized solution is investigated in [7].

In the present paper we propose a simple and natural way of studying generalized solutions.

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Note: Figure translations are in progress. See original paper for figures.