

EXPANSION IN EIGENFUNCTIONS OF NON-SELF-ADJOINT SINGULAR DIFFERENTIAL OPERATORS

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Abstract

Full Text

MATHEMATICS

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**EXPANSION IN EIGENFUNCTIONS OF
NON-SELF-ADJOINT SINGULAR DIFFERENTIAL OPERATORS**

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1. In solving nonstationary problems by the Fourier method, it is necessary to establish the possibility of expanding the initial data in the eigenfunctions of the corresponding boundary-value problems.

The expansion formulas for the boundary-value problem on the half-axis

$$l[y] \equiv y'' - q(x)y = -\lambda^2 y \quad (0 \leq x < \infty), \quad (A)$$

$$y'(0) - Ay(0) = 0, \quad (B)$$

when the problem is self-adjoint (i.e., A and $q(x)$ are real), were first obtained by H. Weyl (¹), who proved the existence of a spectral function $\rho(\lambda)$ (possibly not unique) generating these formulas:

$$E_f(\lambda) = \int_0^\infty f(x)\omega(\lambda, x) dx, \quad f(x) = \int_{-\infty}^\infty E_f(\sqrt{\lambda})\omega(\sqrt{\lambda}, x) d\rho(\lambda) \quad (1)$$

(here $\omega(\lambda, x)$ denotes the solution of equation (A) under the initial conditions $\omega(\lambda, 0) = 1, \omega'(\lambda, 0) = A$).

In the present note we consider an arbitrary non-self-adjoint boundary-value problem (A)–(B) and analogous problems for finite and infinite systems of differential equations.*

2. Let us first consider the boundary-value problem (A)–(B) for a single equation, where $q(x)$ is an arbitrary complex-valued function, summable on every finite interval, and A is any complex number. For this problem we shall give a generalization of Weyl's formulas (1), in which the spectral function $\rho(\lambda)$ turns out already to be a generalized function,** acting in the topological space Z .

Definition. The linear topological space Z is the set of all even entire functions of finite degree (i.e., of exponential type), $F(\lambda)$, summable on the axis $-\infty < \lambda < \infty$.

A sequence $F_n(\lambda) \in Z$ converges to zero if

$$\lim_{n \rightarrow \infty} \int_0^\infty |F_n(\lambda)| d\lambda = 0$$

and the degrees σ_n of the functions $F_n(\lambda)$ are bounded in the aggregate:

$$\sup_n \sigma_n < \infty.$$

* Expansion formulas for non-self-adjoint (regular) problems on a finite interval were obtained by G. Birkhoff and M. V. Keldysh. Special cases of singular non-self-adjoint boundary-value problems were considered in works of M. A. Naimark, B. Ya. Levin, and V. B. Lidskii.

** The concept of a generalized function was first introduced by S. L. Sobolev and systematically developed in recent works by L. Schwartz, I. M. Gelfand—G. E. Shilov, and other authors. In the present note the introduction of generalized functions is closest to the scheme of I. M. Gelfand—G. E. Shilov ⁽²⁾, used by them in the construction of Fourier transforms of rapidly increasing functions.

Continuous linear functionals $T(F)$ acting in the space Z will be called **generalized functions** T , setting $T(F) = (F, T)$, and the set of all generalized functions acting in the space Z will be denoted by $T(Z)$.

Finally, let us agree to denote by W_σ^2 the set of all entire functions of exponential type $\leq \sigma$, square-summable on the real axis.

Theorem 1. Each (in general non-self-adjoint) boundary-value problem (A)—(B) generates a generalized function $R \in T(Z)$ such that

$$(E_f(\lambda)E_g(\lambda), R) = \int_0^\infty f(x)g(x) dx, \quad (2)$$

where $f(x)$ and $g(x)$ are arbitrary finite functions from $L^2(0, \infty)$, and $E_f(\lambda)$ and $E_g(\lambda)$ are their ω -Fourier transforms (1). If $E_f(\lambda) \in L^1(0, \infty)$, then

$$f(x) = (E_f(\lambda)\omega(\lambda, x), R). \quad (3)$$

The generalized function R is called the **spectral function** of the boundary-value problem (A)—(B).

3. Knowing the spectral function R of the problem (A)—(B), one can construct, for the kernel $K(x, y)$ of the corresponding transformation operator

$$\omega(\lambda, x) = \cos \lambda x + \int_0^x K(x, t) \cos \lambda t dt \quad (4)$$

the linear integral equation obtained for self-adjoint problems for the first time by I. M. Gel' fand and B. M. Levitan ⁽³⁾:

$$K(x, y) + f(x, y) + \int_0^x K(x, t) f(t, y) dt = 0 \quad (0 \leq y \leq x), \quad (5)$$

where

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} \left[\left(\frac{\sin \lambda x \sin \lambda y}{\lambda^2}, R \right) - \frac{2}{\pi} \int_0^\infty \frac{\sin \lambda x \sin \lambda y}{\lambda^2} d\lambda \right].$$

Investigation of equation (5) leads to the following generalization, to the non-self-adjoint case, of the corresponding results of I. M. Gel' fand—B. M. Levitan, M. G. Krein ⁽⁴⁾, and one of the authors of this note ⁽⁵⁾.

Theorem 2. In order that a generalized function $R \in T(Z)$ be the spectral function of some boundary-value problem (A)—(B) with an n -times ($n \geq 0$) differentiable function $q(x)$, it is necessary and sufficient that:

1°. The function

$$\Phi(x) = \left(\frac{1 - \cos \lambda x}{\lambda^2}, R \right)$$

have $n + 3$ derivatives for $x \geq 0$, with $\Phi'(0) = 1$.

2°. If $F(\lambda) \in W_\sigma^2$ and $(F(\lambda)X(\lambda), R) = 0$ for all $X(\lambda) \in W_\sigma^2$, then $F(\lambda) \equiv 0$.

In this case $q(x)$ and A are determined uniquely by R .

4. The proposed method generalizes to boundary-value problems for finite and infinite systems of differential equations. These problems are described by equation (A) and the boundary condition (B), if one assumes that $q(x)$ for each $x \geq 0$ and A are arbitrary bounded operators in some Banach space B , with $q(x)$ depending continuously on x . In addition to the solutions $\omega(\lambda, x)$, which now already are operator-functions, we shall also need solutions $\tilde{\omega}(\lambda, x)$ of the equation

$$v'' - vq(x) + \lambda^2 v = 0$$

with initial data $v(0) = I$, $v'(0) = A$, and the $\tilde{\omega}$ -transform

$$\tilde{E}_f(\lambda) = \int_0^\infty \tilde{\omega}(\lambda, x) f(x) dx.$$

For simplicity taking B to be a separable Hilbert space, we may write the operator-functions that occur in matrix form:

$$q(x) = \|q_{ik}(x)\|, \quad \omega(\lambda, x) = \|\omega_{ik}(\lambda, x)\|,$$

and so on.

Theorem 3. Every operator boundary-value problem (A)–(B) generates a spectral matrix $R = \|R_{ik}\|$ with elements $R_{ik} \in T(Z)$ such that

$$(E_f(\lambda)R\tilde{E}_g(\lambda)) = \int_0^\infty f(x)g(x) dx, \quad (6)$$

where $f(x)$ and $g(x)$ are continuous finite operator-functions; $E_f(\lambda)$ and $E_g(\lambda)$ are their ω - and $\tilde{\omega}$ -transforms, respectively;

$$(E_f(\lambda)R\tilde{E}_g(\lambda)) = \left\| (E_f^l(\lambda)\tilde{E}_g^k(\lambda), R_{lk}) \right\|$$

(summation is performed over identical indices).

The investigation of the inverse problem again leads to equation (5), this time for operator-functions, from which the following theorem is derived.

Theorem 4. In order that a matrix R with elements $R_{ik} \in T(Z)$ be the spectral matrix of some operator boundary-value problem (A)–(B) with an n -times ($n \geq 0$) continuously differentiable operator-function $q(x)$, it is necessary and sufficient that the following conditions hold:

1°. The operator-function

$$\Phi(x) = \left(\frac{1 - \cos \lambda x}{\lambda^2}, R \right)$$

for $x \geq 0$ has $n + 3$ continuous derivatives and $\Phi'(+0) = I$.

2°. If the vector-function

$$\mathbf{h}(\lambda) = (h_1(\lambda), h_2(\lambda), \dots)$$

belongs to W_σ^2 (i.e. $h_i(\lambda) \in W_\sigma^2$ and

$$|\mathbf{h}(\lambda)|^2 = \int_0^\infty \sum_{i=1}^\infty |h_i(\lambda)|^2 d\lambda < \infty),$$

then

$$\sup |(\mathbf{x}(\lambda)R\mathbf{h}(\lambda))| \geq \varepsilon(\sigma)|\mathbf{h}(\lambda)|, \quad \sup |(\mathbf{h}(\lambda)R\mathbf{x}(\lambda))| \geq \varepsilon(\sigma)|\mathbf{h}(\lambda)|,$$

where the supremum is taken over all $\mathbf{x}(\lambda) \in W_\sigma^2$ for which $|\mathbf{x}(\lambda)| = 1$,

$$(\mathbf{a}(\lambda)R\mathbf{b}(\lambda)) = \sum_{i,k} (a_i(\lambda)b_k(\lambda), R_{ik}),$$

and $\varepsilon(\sigma) > 0$ does not depend on $\mathbf{h}(\lambda)$.

5. The multipliers in the space Z are the characteristic functions of compact sets bounded on the real axis.

Therefore the formulas obtained make it possible to solve nonstationary problems only for equations $l_x[u] = u_{tt}$. However, the spectral function R of each concrete problem can always be extended (in general, not uniquely) to some space broader than Z , and the finding of such extensions constitutes the content of the spectral analysis of the given boundary-value problem. These extensions are connected with the notion of the spectrum of the problem and with the analytic form of the spectral function R . At the same time, if, for example, the functions $e^{-\lambda^2 t}$, $e^{i\lambda^2 t}$, and so forth turn out to be multipliers in the extended space, then this will make it possible to solve, by the Fourier method, equations of the form $l_x[u] = u_t$, $l_x[u] = iu_t$, and so on.

Without dwelling here on particular examples of such extensions, we shall consider only the case of a self-adjoint operator boundary-value problem. In this case the spectral matrix R is positive in the sense that if $F(\lambda) \in Z$ and $F(\sqrt{\lambda}) \geq 0$ for $-\infty < \lambda < \infty$, then

$$(F(\lambda), R) = \|(F(\lambda), R_{ik})\|$$

is the matrix of a positive operator. This makes it possible to apply M. Riesz' s method of extending a positive functional ⁽⁶⁾, by means of which Theorem 5 is proved.

Theorem 5. If $q^*(x) = q(x)$ and $A^* = A$, then the spectral matrix R is an operator measure, i.e.

$$\int_0^\infty f(x)g^*(x) dx = (E_f(\lambda)RE_g^*(\bar{\lambda})) = \int_{-\infty}^\infty E_f(\sqrt{\lambda}) [d\rho(\lambda)] E_g^*(\sqrt{\lambda}), \quad (7)$$

where $\rho(\lambda)$ is a nondecreasing operator-function ($\rho(\lambda + h) - \rho(\lambda)$ for $h \geq 0$ is a nonnegative bounded operator).

For the operator-function $\rho(\lambda)$ there is an asymptotic formula analogous to the scalar case (7).

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Note: Figure translations are in progress. See original paper for figures.

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