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Abstract

Full Text

MATHEMATICS

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IRREGULAR SURFACES OF BOUNDED EXTERNAL CURVATURE

(Presented by Academician V. I. Smirnov on 18 XI 1957)

1. The intrinsic geometry of abstract irregular surfaces created by A. D. Aleksandrov* (the theory of manifolds with an intrinsic metric of bounded curvature) has led to the study of a number of classes of surfaces with various restrictions on the character of the embedding of the surface in three-dimensional Euclidean space E^3 , which ensure boundedness of the curvature of the intrinsic metric of the surface ^(2,3,4).

Restricting himself to the requirement of smoothness of surfaces*, A. V. Pogorelov ⁽⁴⁾ introduced and investigated the class of surfaces of bounded external curvature. Surfaces of this class are characterized by the fact that they have finite area of the spherical image, counted with multiplicity. This class includes all previously studied classes of smooth surfaces.

However, the requirement of smoothness of a surface is very strong. It excludes from consideration general convex surfaces and DC surfaces (representable as differences of convex functions ⁽²⁾). In this connection A. D. Aleksandrov posed the problem of singling out a class of surfaces with intrinsic metric of bounded curvature which, along with the aforementioned smooth surfaces of bounded external curvature, would include general convex surfaces and DC surfaces. The present work is devoted to the solution of this problem.

2. We shall consider surfaces F satisfying the following conditions:
 - 1) At each point X of the surface F , the contingency of the surface is a cone $K_F(X)$. We shall call it the **tangent cone of the surface at the given point**. Suppose, further, that a sequence of points X_1, X_2, \dots of the surface F converges to the point $X_0 \in F$, and P_1, P_2, \dots is a convergent sequence of tangent planes to the cones $K_F(X_1), K_F(X_2), \dots$. Then the limiting plane P_0 is a tangent plane to the cone $K_F(X_0)$.
 - 2) Each point X of the surface F has a neighborhood U which, in suitably chosen Cartesian coordinates, is representable by the equation $z = f(x, y)$, where the tangent cones at the points of U have no tangent planes perpendicular to the x, y -plane.

Conditions 1) and 2) replace the requirement of smoothness of the surface F by weaker restrictions, which are satisfied for general convex surfaces and DC surfaces; moreover, they make it possible to establish that on the surface F there is a sufficient number of rectifiable curves in order to introduce on F an intrinsic metric in the known way ⁽¹⁾.

Without loss of generality, one may further assume that the surface F is given by the equation $z = f(x, y)$, and that the tangent planes to its tangent cones make with the x, y -plane angles not exceeding $\pi/2 - \delta$, where $0 < \delta < \pi/2$.

* A surface in E^3 is called smooth if, in a neighborhood of each of its points, it admits representation by an equation $z = f(x, y)$, where $f(x, y)$ is a function continuous together with its first derivatives.

Let us pass to the conditions that bound the extrinsic curvature of the surface. In the case under consideration, it is rather difficult to use the generalized spherical image of a surface. It is more convenient to bound the positive part of the extrinsic curvature in the following way. Let $G \subset F$ be a closed compact domain, and let $M_1, M_2, \dots, M_n \subset G$ be open pairwise disjoint sets with convex projections $\widetilde{M}_1, \widetilde{M}_2, \dots, \widetilde{M}_n$ onto the x, y -plane. Let $\overline{M}_1, \overline{M}_2, \dots, \overline{M}_n$ be the convex hulls of the sets M_1, M_2, \dots, M_n . Denote by M'_i and M''_i those parts of the boundary of the convex body \overline{M}_i whose convexity is directed respectively toward $z > 0$ and $z < 0$, and which are projected one-to-one onto the open set \widetilde{M}_i , and let, respectively, σ'_i and σ''_i be the areas of their spherical images. Consider

$$\sigma(G) = \sup \sum_{i=1}^n (\sigma'_i + \sigma''_i),$$

where the least upper bound is taken over all possible systems of the sets M_1, M_2, \dots, M_n introduced above. The quantity $\sigma(G)$ determines the positive part of the extrinsic curvature of the domain G of the surface F . If the surface F satisfies conditions 1), 2) and the quantity $\sigma(G)$ is finite for every domain $G \subset F$ that is compact in the sense of the intrinsic metric, then it is natural to call F a surface of bounded extrinsic curvature.

If, however, one assumes that the surface F is given by a single equation $z = f(x, y)$, then, using conditions 1) and 2), the requirement of boundedness of the positive part of the extrinsic curvature may be imposed locally. By **surfaces of bounded extrinsic curvature** we shall understand, in what follows, surfaces satisfying conditions 1), 2) for which the positive part of the extrinsic curvature is locally bounded.

It is easy to verify that the class of surfaces of bounded extrinsic curvature includes the classes of general convex surfaces, surfaces of bounded turn, and smooth surfaces of bounded extrinsic curvature.

3. Theorem 1. For every point X of a surface F of bounded extrinsic curvature there exists a neighborhood $U \subset F$ such that there is a sequence of regular surfaces F_n converging uniformly to F in U together with the intrinsic metrics, and the positive parts of whose extrinsic curvatures

$$\iint_{E_n} K_n dS_n$$

are uniformly bounded (E_n is the set of points of F_n where the Gaussian curvature $K_n \geq 0$; dS_n is the area element of the surface F_n).

The construction of the regular surfaces F_n is carried out with the aid of the constructions proposed by A. V. Pogorelov in ⁽⁴⁾.

From Theorem 1 it follows easily:

Theorem 2. Surfaces of bounded extrinsic curvature in the sense of their intrinsic metric are manifolds of bounded curvature in the sense of A. D. Aleksandrov.

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CITED LITERATURE

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- ⁴ A. V. **Pogorelov**, *Surfaces of Bounded Extrinsic Curvature*, Kharkov, 1956.

Note: Figure translations are in progress. See original paper for figures.

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