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## Abstract

## Full Text

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## MATHEMATICS

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# SOLUTION OF A SYSTEM OF FINITE-DIFFERENCE EQUATIONS CORRESPONDING TO THE DIRICHLET PROBLEM BY MEANS OF A NORMAL SEQUENCE OF SIGNS

(Presented by Academician I. M. Vinogradov on 26 IX 1958)

Let  $g \geq 2$  be natural. Suppose there is an infinite sequence composed of the signs  $0, 1, 2, \dots, g-1$ :

$$\varepsilon_1, \varepsilon_2, \dots \quad (1)$$

Given a natural number  $s \geq 1$ , write the first  $P + s - 1$  signs of sequence (1) in the following form:

$$(\varepsilon_1 \varepsilon_2 \dots \varepsilon_s)(\varepsilon_2 \varepsilon_3 \dots \varepsilon_{s+1}) \dots (\varepsilon_P \varepsilon_{P+1} \dots \varepsilon_{P+s-1}) \quad (2)$$

and call the sequence of brackets (2) the **caterpillar of length  $P$**  (of rank  $s$ ) of sequence (1). Let  $\Delta = (\delta_1 \dots \delta_s)$  be some group composed of  $s$  signs  $0, 1, \dots, g-1$ . Denote by  $N_P(\Delta)$  the number of times  $\Delta$  occurs in the caterpillar of length  $P$ .

We shall call sequence (1) a **normal sequence of signs** if, whatever  $s \geq 1$  we take and whatever group  $\Delta$  we prescribe,

$$\lim_{P \rightarrow \infty} \frac{N_P(\Delta)}{P} = \frac{1}{g^s}. \quad (3)$$

It is easy to show that if  $\alpha$  is a number of the interval  $[0, 1]$  such that the fractional parts  $\{\alpha g^x\}$  are uniformly distributed, and if  $\alpha = 0, \varepsilon_1 \varepsilon_2 \dots$  is the  $g$ -adic expansion of  $\alpha$ , then the sequence  $\varepsilon_1, \varepsilon_2, \dots$  is normal. Conversely, if sequence (1) is normal, then

$$\alpha = \frac{\varepsilon_1}{g} + \frac{\varepsilon_2}{g^2} + \dots$$

has the property that  $\{\alpha g^x\}$  are uniformly distributed.

There exist many methods of constructing normal sequences of signs <sup>(1,2)</sup>.

Denote  $\mu\Delta = 1/g^s$  and call  $\mu\Delta$  the **measure of the combination**  $\Delta$ . Consider a set  $\mathfrak{M}$  of various combinations  $\Delta$ , in which there may be combinations of different lengths (different  $s$ ), but combinations of infinite length do not enter into  $\mathfrak{M}$ . By the measure  $\mu\mathfrak{M}$  of  $\mathfrak{M}$  we shall mean the sum of the measures of the combinations entering into it (if this sum is finite).

We shall consider sets  $\mathfrak{M}$  satisfying the following two requirements:

1. The set  $\mathfrak{M}$  is such that if a combination  $(\delta_1\delta_2 \dots \delta_s)$  belongs to it, then the combinations  $(\delta_1)$ ,  $(\delta_1\delta_2)$ , ...,  $(\delta_1\delta_2 \dots \delta_{s-1})$  do not belong to it.
2. Let  $k$  be an integer. In the combinations of  $\mathfrak{M}$  with a number of signs greater than or equal to  $k + 1$ , leave the first  $k$  signs. We obtain a set of combinations, which we denote by  $\overline{\mathfrak{M}}_{(k)}$ .

Suppose that

$$\lim_{k \rightarrow \infty} \mu\overline{\mathfrak{M}}_{(k)} = 0.$$

Take in some normal sequence of signs (1) the  $k$ -th place and form the combinations  $(\varepsilon_k)$ ,  $(\varepsilon_k\varepsilon_{k+1})$ ,  $(\varepsilon_k\varepsilon_{k+1}\varepsilon_{k+2})$  ... until it turns out that  $(\varepsilon_k\varepsilon_{k+1} \dots \varepsilon_l)$  belongs to  $\mathfrak{M}$ . In that and only in that case, if this happens, we shall say that at the  $k$ -th place of the sequence (1) a combination from  $\mathfrak{M}$  has appeared. Denote by  $N_P(\mathfrak{M})$  the number of appearances of combinations from  $\mathfrak{M}$  up to the  $P$ -th place of the sequence (1).

**Theorem.** *If the sequence (1) is normal, and  $\mathfrak{M}$  satisfies the assumptions made above, then*

$$\lim_{P \rightarrow \infty} \frac{N_P(\mathfrak{M})}{P} = \mu\mathfrak{M}.$$

**Proof.** Denote by  $\mathfrak{M}^{(s)}$  the set of combinations from  $\mathfrak{M}$  consisting of  $s$  signs. It is clear that for any  $k$

$$\sum_{s=1}^k N_P(\mathfrak{M}^{(s)}) \leq N_P(\mathfrak{M}) \leq \sum_{s=1}^k N_P(\mathfrak{M}^{(s)}) + N_P(\overline{\mathfrak{M}}_{(k)}).$$

Hence, by virtue of the normality of the sequence (1),

Fig. 1

Figure 1: Fig. 1

$$\sum_{s=1}^k \mu \mathfrak{M}^{(s)} \leq \underline{\lim}_{P \rightarrow \infty} \frac{N_P(\mathfrak{M})}{P} \leq \overline{\lim}_{P \rightarrow \infty} \frac{N_P(\mathfrak{M})}{P} \leq \sum_{s=1}^k \mu \mathfrak{M}^{(s)} + \mu \overline{\mathfrak{M}}_{(k)}.$$

Since  $\mathfrak{M}$  contains no infinite combinations, then

$$\lim_{k \rightarrow \infty} \sum_{s=1}^k \mu \mathfrak{M}^{(s)} = \mu \mathfrak{M}.$$

By the condition of the theorem,

$$\lim_{k \rightarrow \infty} \mu \overline{\mathfrak{M}}_{(k)} = 0.$$

Letting  $k$  tend to infinity, we obtain

$$\lim_{P \rightarrow \infty} \frac{N_P(\mathfrak{M})}{P} = \mu \mathfrak{M},$$

as was required.

In solving the Dirichlet problem in the plane <sup>(3)</sup> one has to solve the following system of linear equations. Suppose that in the plane a bounded simply connected domain  $D$  is given and that there is a square grid (see Fig. 1). We shall call a grid point a **boundary** point if at least one of its four neighboring points does not belong to  $D$ , and an **interior** point if all four neighbors belong to  $D$ .

**Fig. 1**

Suppose that at the boundary points some numbers  $\lambda_1, \lambda_2, \dots, \lambda_s$  are prescribed. Consider the system of linear equations:

$$u(x, y) = \frac{1}{4}[u(x+1, y) + u(x-1, y) + u(x, y+1) + u(x, y-1)],$$

if  $(x, y)$  is an interior point;

$$u(x, y) = \lambda_i, \quad \text{if } (x, y) \text{ is a boundary point.} \quad (4)$$

The number of unknowns is equal to the number of interior points. We wish to determine  $u(x, y)$  at some fixed interior point. For this purpose we take a normal sequence composed of the signs 0, 1, 2, 3:

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots$$

Suppose a particle moves along the grid points; let the sign 0 mean an order to the particle to move one unit to the left, the sign 1 an order to move one unit upward, the sign 2—to the right, the sign 3—downward; let the boundary points be absorbing points, i.e., once the particle reaches a boundary point, it remains there regardless of any orders. Fix some boundary-

boundary point  $B_i$  and denote by  $\mathfrak{M}_{B_i}$  the set of such combinations  $(\delta_1 \delta_2 \dots)$  that the particle, which was initially at the node  $(x, y)$ , following these orders, after the last order will be absorbed at the point  $B_i$ . It is clear that the first requirement on  $\mathfrak{M}$  is satisfied. Obviously,  $\mu \mathfrak{M}$  is equal to the probability that a particle which begins random wandering at the point  $(x, y)$  and can move each time in all four directions with probability 1/4 will eventually be absorbed at the point  $B_i$ . Denote

$$u_i(x, y) = \mu \mathfrak{M}_{B_i}.$$

In Feller' s book ((<sup>4</sup>, p. 307)) it is shown that  $u_i(x, y)$  satisfies the system of linear equations

$$\begin{aligned} u_i(x, y) &= \frac{1}{4}(u_i(x+1, y) + u_i(x-1, y) + u_i(x, y+1) + u_i(x, y-1)), \\ &\quad \text{if } (x, y) \text{ is an interior point;} \\ u_i(x, y) &= 1, \quad \text{if } (x, y) = B_i; \\ u_i(x, y) &= 0, \quad \text{if } (x, y) \text{ is a boundary point distinct from } B_i. \end{aligned}$$

Further, in Feller' s book it is shown that the probability of "eternal wandering" is equal to zero, i.e.

$$\lim_{k \rightarrow \infty} \overline{\mathfrak{M}}(k) = 0$$

( $\overline{\mathfrak{M}}_{B_i(k)}$  does not depend on  $B_i$ ). Applying the theorem, we obtain

$$\lim_{P \rightarrow \infty} \frac{N_P(\mathfrak{M}_{B_i})}{P} = \mu \mathfrak{M}_{B_i}.$$

Further, it is obvious that

$$u(x, y) = \sum_{i=1}^s \lambda_i \mu \mathfrak{M}_{B_i},$$

i.e.

$$u(x, y) = \lim_{P \rightarrow \infty} \sum_{i=1}^s \lambda_i \frac{N_P(\mathfrak{M}_{B_i})}{P}. \quad (5)$$

We carry out motions from the point  $(x, y)$  according to the orders

$$(\varepsilon_1 \varepsilon_2 \dots)(\varepsilon_2 \dots)(\varepsilon_3 \dots) \dots (\varepsilon_P \dots) \dots$$

and count how many times absorptions take place at the points  $B_1, B_2, \dots, B_s$ , i.e. determine the quantities  $N_P(\mathfrak{M}_{B_i})$ ,  $i = 1, 2, \dots, s$ . Formula (5) gives the solution of system (4).

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*Note: Figure translations are in progress. See original paper for figures.*

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