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Abstract

Full Text

PHYSICS

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ON THE EQUILIBRIUM FUNCTION OF THE ANGULAR DISTRIBUTION OF PARTICLES IN A CASCADE SHOWER

(Presented by Academician D. V. Skobel' tsyn, 20 V 1958)

Many authors' works have been devoted to determining the distribution function of electrons at the shower maximum [(1–5)]; however, the problem still cannot be regarded as definitively solved. Work [(1)] has repeatedly been criticized by many authors (see, for example, [(2)]). In works [(3–5)] the problem was solved without taking ionization losses into account. In work [(2)] the problem posed was solved most fully; however, the numerical method of solution used by the authors is so complicated that it is impossible to estimate the accuracy of the results obtained. In all the works mentioned, the small-angle approximation was used, scattering was treated as multiple scattering, and the problem was solved only for a primary particle of infinite energy.

In the present work, in the approximation of multiple scattering, formulas are obtained for the angular distribution function of particles at the shower maximum, with allowance for ionization losses, in the approximation of small and large angles relative to a primary particle of infinite energy. Formulas for the required function are also obtained in the case of a shower from a primary particle of finite energy.

We write the initial equations for the distribution function of electrons $P(E_0, E, \vartheta)$ with respect to energy and angles in the case of a primary electron of energy E_0, vertically incident on the boundary of a layer of matter at $t = 0$, in the form

$$-\frac{\cos\vartheta\,\delta(E_0 - E)\delta(\vartheta)}{2\pi} = \int_E^\infty P_p(E_0, E', \vartheta)\varphi(E', E)\,dE' - \mu_e(E)P_p(E_0, E, \vartheta) +$$

$$+\beta\,\frac{\partial P_p(E_0, E, \vartheta)}{\partial E} + \frac{E_k^2}{4E^2}\Delta_\vartheta P_p(E_0, E, \vartheta); \quad (1)$$

$$\varphi(E', E) = 2\varphi'(E', E) + W_e(E', E' - E); \qquad \mu_e(E) = \int_0^E W_e(E, E')\,dE';$$

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$$\varphi'(E', E) = \int_E^{E'} \frac{W_p(E'', E) W_e(E'', E)}{\sigma(E'')} \, dE''.$$

Here W_e and W_p are the probabilities of bremsstrahlung and pair-production processes, $\sigma(E)$ is the total photon absorption coefficient, and Δ_ϑ is the Laplace operator. The corresponding photon distribution function is determined by the expression

$$\Gamma_p(E_0, E, \vartheta) = \frac{1}{\sigma(E)} \int_E^\infty P_p(E_0, E', \vartheta) W_e(E', E) \, dE'. \tag{2}$$

For the time being we shall assume that the angles under consideration are $\vartheta \ll 1$, i.e., in (1) we replace $\sin\vartheta$ by ϑ, and $\cos\vartheta$ by 1. Multiplying equation (1) by $J_0(\lambda\vartheta)\vartheta$ and integrating with respect to ϑ from 0 to ∞, as a result we obtain:

$$\left[L + \beta \frac{\partial}{\partial E}\right] f(E_0, E, \lambda) = \frac{\delta(E_0 - E)}{2\pi}. \tag{3}$$

Here

$$f(E_0, E, \lambda) = \frac{\partial n(E_0, E, \lambda)}{\partial E};$$

$$n(E_0, E, \lambda) = \int_0^\infty N_p(E_0, E, \vartheta) J_0(\lambda\vartheta)\vartheta \, d\vartheta;$$

$$N_p(E_0, E, \vartheta) = \int_E^\infty P_p(E_0, E', \vartheta) \, dE'.$$

The operator L is defined by the equality

$$Lf(E_0, E, \lambda) = \int_E^\infty f(E_0, E', \lambda)\varphi(E', E) \, dE' - \left\{\mu_e(E) + \frac{\alpha}{E^2}\lambda^2\right\} f(E_0, E, \lambda);$$

$$\alpha = \frac{E_k^2}{4}; \qquad E_k = 21 \text{ MeV}.$$

Let us introduce an arbitrary function U, satisfying the condition $U(E, E_1, \lambda) = 0$ for $E < E_1$. We define the adjoint operator L^*. Then, multiplying (3) by $U(E, E_1, \lambda)$ and integrating with respect to E from 0 to ∞, we obtain

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$$\int_0^\infty f(E_0, E, \lambda) \left[L^* - \beta \frac{\partial}{\partial E}\right] U(E, E_1, \lambda)\, dE = \frac{U(E_0, E_1, \lambda)}{2\pi}. \qquad (4)$$

If the function U is given, then (4) can be regarded as an integral equation for the function $f(E_0, E, \lambda)$. Suppose that $U(E, E_1, \lambda)$ satisfies the equation

$$\left[L^* - \beta \frac{\partial}{\partial E}\right] U(E, E_1, \lambda) = \frac{\delta(E - E_1)}{2\pi}. \qquad (5)$$

Then it follows from (4) that $U(E_0, E_1, \lambda) = f(E_0, E_1, \lambda)$, i.e., solving the original equation (3) is equivalent to solving (4). One can use the adjoint equation (5) to obtain an equation satisfied by the function $n(E_0, E, \lambda)$. To this end we integrate (5) with respect to E_1 from E_2 to ∞. After some transformations we obtain

$$\left[L^* - \beta \frac{\partial}{\partial E_0}\right] n(E_0, E, \lambda) = -\frac{1}{2\pi}. \qquad (6)$$

Let us now consider the equation

$$L^* n(E_0, E, \lambda) = -\frac{1}{2\pi}. \qquad (7)$$

This is the equation for the transformed function $N(E_0, E, \vartheta)$ without allowance for ionization losses. The solution of (7) was given in work ([5]):

$$n(E, E_1, \lambda) = 0 \qquad \text{for } E < E_1;$$

$$n(E, E_1, \lambda) = \frac{E^2}{4\pi q} \left\{(E^2 + p^2\lambda^2)(E_1^2 + p^2\lambda^2)\right\}^{-1/2} \qquad \text{for } E > E_1; \qquad p^2 = \frac{\alpha}{q}.$$

The function $n(E, E_1, \lambda)$ has a discontinuity at $E = E_1$, therefore (for $E^2 \gg p^2\lambda^2$)

$$\frac{\partial n(E, E_1, \lambda)}{\partial E} = \frac{1}{2\pi q}(E_1^2 + p^2\lambda^2)^{-1/2} + \frac{1}{2\pi q} E_1 \delta(E - E_1)(E_1^2 + p^2\lambda^2)^{-1/2}. \quad (8)$$

Taking (8) into account, equation (4) can be written in the form

$$\frac{\partial n(E_0, E, \lambda)}{\partial E} - n(E_0, E, \lambda) \left[\frac{q(E^2 + p^2\lambda^2)^{1/2}}{\beta E} + \frac{1}{E}\right] + \frac{E_0}{2\pi\beta E} = 0. \quad (4')$$

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Its solution is not difficult to find:

$$n(E_0, E, \lambda) = \frac{1}{2\pi}\frac{E_0}{\beta}E^{qp\lambda/\beta+1}\{p\lambda + (E^2 + p^2\lambda^2)^{1/2}\}^{-qp\lambda/\beta}\exp\{q(E^2 + p^2\lambda^2)^{1/2}/\beta\}$$
$$\times \int_E^{E_0} \exp\{-q(x^2 + p^2\lambda^2)^{1/2}/\beta\}\{p\lambda + (x^2 + p^2\lambda^2)^{1/2}\}^{qp\lambda/\beta} x^{-qp\lambda/\beta-2}\, dx, \tag{9}$$

whence, using the inversion formula for the Fourier–Bessel transform, we obtain in the limit as $E = 0$ an explicit expression for the function $N_p(E_0, 0, \vartheta)$:

$$N_p(E_0, 0, \vartheta) = \frac{1}{2\pi}\frac{E_0}{qp}\frac{1}{\vartheta}\left\{1 - \frac{\pi}{2}\frac{\vartheta\beta}{qp}\left[H_0\left(\frac{\vartheta\beta}{qp}\right) - Y_0\left(\frac{\vartheta\beta}{qp}\right)\right]\right\}. \tag{10}$$

Hence, for $\vartheta \ll qp/\beta$,

$$N_p(E_0, 0, \vartheta) = \frac{1}{2\pi}\frac{E_0}{qp}\frac{1}{\vartheta} \tag{11}$$

and for $\vartheta \gg qp/\beta$,

$$N_p(E_0, 0, \vartheta) = \frac{1}{2\pi}\frac{E_0}{\beta}\frac{qp}{\beta}\frac{1}{\vartheta^3}. \tag{12}$$

In work ([6]) the solution of equation (1) was obtained in the form of a series in Legendre polynomials

$$N_p(E_0, E, \vartheta) = \sum_{n=0}^{\infty} f_n^p(E_0, E) P_n(\cos\vartheta);$$

$$f_n^p(E_0, E) = \frac{2n+1}{4\pi}\frac{\varepsilon_0}{q}\varepsilon^{a_n+1}\{a_n + (\varepsilon^2 + a_n^2)^{1/2}\}^{a_n}\exp\{(\varepsilon^2 + a_n^2)^{1/2}\}$$
$$\times \int_\varepsilon^{\varepsilon_0} \exp\{-(x^2 + a_n^2)^{1/2}\}\{a_n + (x^2 + a_n^2)^{1/2}\}^{a_n} x^{-a_n-2}\, dx.$$

Here

$$a_n = \frac{1}{2}\varepsilon_k\{n(n+1)/q\}^{1/2}, \qquad \varepsilon = Eq/\beta.$$

Using explicit expressions for the sums of the series $\sum_{n=0}^{\infty} P_n(x)$ and $\sum_{n=1}^{\infty} P_n(x)/n$, in the limit as $E = 0$ we obtain an expression for the function $N_p(E_0, 0, \vartheta)$:

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$$N_p(\varepsilon_0, 0, \vartheta) = \frac{\varepsilon_0}{2\pi\varepsilon_k\sqrt{q}}\frac{1}{\sin(\vartheta/2)} + \frac{2\varepsilon_0}{\pi\varepsilon_k^2}\left\{\ln\sin\frac{\vartheta}{2} + \ln\left(1+\sin\frac{\vartheta}{2}\right)\right.$$
$$+\frac{\varepsilon_0}{4\pi q}\sum_{n=1}^{\infty}\frac{2-2(1-1/\alpha n)(1+1/n)^{1/2}+(1-2/\alpha)/n+2/\alpha^2n^2}{\alpha(1+1/n)^{1/2}+1/n}\left(\frac{2}{2n+1}\right)^{1/2}\overline{P}_n(x)$$
$$\left.+\frac{1}{\pi}\frac{\varepsilon_0}{q}\left(\frac{1}{4}-\frac{\sqrt{q}}{\varepsilon_k}\right)\right\}; \qquad \alpha = \frac{\varepsilon_k}{2\sqrt{q}}. \tag{13}$$

The remaining series* converges absolutely faster than $n^{-2.5}$. Let $\vartheta \ll 1$; then from (13) we obtain

$$N_p(E_0, 0, \vartheta) = \frac{E_0}{2\pi pq}\frac{1}{\vartheta}. \tag{14}$$

For the angular-distribution function in the large-angle approximation, derived in ([5]) without taking ionization losses into account and for $E_0 = \infty$, we obtain

* From the remaining series one can separate out $\sum_{n=1}^{\infty} P_n(x)/n^2$, etc., which are expressed in terms of integrals of the generating function.

$$N_p(E_0, E, \vartheta) = \frac{E_0}{4\pi\rho q}\left\{\frac{1}{\sin(\vartheta/2)} + \sum_{n=1}^{\infty}\frac{2n+1-2n(1+1/n+k^2/n^2)^{1/2}}{n(1+1/n+k^2/n^2)^{1/2}}\left(\frac{2}{2n+1}\right)^{1/2}\overline{P}_n(x)\right\} + \frac{E_0}{4\pi q}\left(\frac{1}{E}-\frac{2}{p}\right) \tag{15}$$

The remaining series converges absolutely faster than $n^{-2.5}$. For $\vartheta \ll 1$,

$$N_p(E_0, E, \vartheta) = \frac{E_0}{2\pi\rho q}\frac{1}{\vartheta}, \tag{16}$$

which coincides with the expression obtained in (5) in the small-angle approximation.

For the corresponding function derived for finite E_0 (5), we obtain

$$N_p(E_0, E, \vartheta) = \frac{E_0^2}{4\pi q\rho^2}\left\{-2\ln\sin\frac{\vartheta}{2} - 2\ln\left(1+\sin\frac{\vartheta}{2}\right) + \right.$$

$$\left.+\sum_{n=1}^{\infty}\left(\frac{2}{2n+1}\right)^{1/2}\overline{P}_n(x)\frac{2n+1-2n\{(1+1/n+k_1^2/n^2)(1+1/n+k^2/n^2)\}^{1/2}}{n^2\{(1+1/n+k_1^2/n^2)(1+1/n+k^2/n^2)\}^{1/2}}\right\} + \frac{E_0}{E}\frac{1}{4\pi q}. \tag{17}$$

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Fig. 1. Functions of the angular distribution of electrons at shower maximum.

Figure 1: Fig. 1. Functions of the angular distribution of electrons at shower maximum.

$$k_1^2 = E_0^2 p^2.$$

The remaining series converges absolutely faster than $n^{-2.5}$. For $\vartheta \ll 1$ we obtain:

$$N_p(E_0, E, \vartheta) = \frac{E_0^2}{2\pi q \rho^2} \left\{ -\ln \frac{\vartheta}{2} \right\}. \tag{18}$$

Similarly, in the small-angle approximation

$$N_p(E_0, E, \vartheta) = \frac{E_0^2}{4\pi q \rho^2} \int_0^\infty \frac{J_0(u)\, u\, du}{(z_0^2 + u^2)^{1/2} (z^2 + u^2)^{1/2}};$$

$$z = \frac{E\vartheta}{p}; \qquad z_0 = \frac{E_0 \vartheta}{p}.$$

It is known that $\vartheta \to 0$ corresponds to $u \to \infty$. Therefore, expanding $(z_0^2 + u^2)^{-1/2}$ in powers of u^{-1}, we obtain

$$N_p(E_0, E, \vartheta)_{\vartheta \to 0} = \frac{E_0^2}{2\pi q \rho^2} \left\{ -\ln \frac{\vartheta}{2} \right\}. \tag{19}$$

Fig. 1. Functions of the angular distribution of electrons at shower maximum. 1 —total number of particles in the small-angle approximation; 2 —the same in the large-angle approximation; 3 —number of particles with energy greater than E in the large-angle approximation. 1, 2, 3 —$E_0 = \infty$; 4 —the same as 3, but for finite E_0, $K_1 = 15$, $K = 2$.

Thus, formulas (10), (13), (15) are strictly applicable only for $E_0 = \infty$. For any finite value of E_0, the character of the behavior of $N_p(E_0, E, \vartheta)$ changes sharply in comparison with $E_0 = \infty$. The results of the calculation of the angular-distribution function are presented in Fig. 1.

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References

1. G. Moliere, *Vorträge über kosmische Strahlung*, Berlin, 1953, p. 446.
2. J. Nishimura, K. Kamata, *Progr. Theor. Phys.*, **6**, 262 (1951).
3. M. H. Kalos, J. M. Blatt, *Austral. J. Phys.*, **7**, 543 (1954).
4. B. A. Chartress, H. Messel, *Phys. Rev.*, **99**, 1604 (1955).
5. S. Z. Belenky, *ZhETF*, **15**, 7 (1945).
6. I. P. Ivanenko, *ZhETF*, **32**, 333 (1957).

Note: Figure translations are in progress. See original paper for figures.

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