MATHEMATICS

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

S. V. YANOVSKII

ON THE CONNECTION BETWEEN INTEGRAL EQUATIONS OF CONVOLUTION TYPE AND EQUATIONS WITH A CAUCHY KERNEL

(Presented by Academician N. I. Muskhelishvili, 4 XI 1957)

As was shown in ([1]), the integral equation of convolution type

$$f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_i(x-t) f(t)\, dt + \int_{-\infty}^{\infty} n(x;-t) f(t)\, dt = g(x), \qquad (1)$$

$$i = 1 \text{ for } t > 0; \qquad i = 2 \text{ for } t < 0; \qquad -\infty < x < \infty,$$

is reduced by the Fourier transform to a certain equivalent singular integral equation with Cauchy kernel

$$A(\zeta)\Phi(\zeta) + \frac{B(\zeta)}{\pi i} \int_\Gamma \frac{\Phi(\tau)\, d\tau}{\tau - \zeta} + \int_\Gamma M(\zeta,\tau)\Phi(\tau)\, d\tau = Q(\zeta), \qquad \zeta \in \Gamma. \quad (2)$$

The unknown function $f(x)$ is assumed to belong to $L_2(-\infty,\infty)$,
$k_i(x) \in L(-\infty,\infty)$, $n(x,t) \in L_2\binom{-\infty,\infty}{-\infty,\infty}$, $g(x) \in L_2(-\infty,\infty)$.

Under broader assumptions concerning the kernels or the unknown function, in subsequent works ([2–5]) the characteristic equation ($n(x,t) \equiv 0$) was completely studied and the solution was given in closed form. For one special case in ([4]) the complete equation (1) was also considered and its connection with equation (2) was established. In the present note this question is solved in the general case and with a certain weakening of the restrictions imposed in ([2–5]) on the kernels $k_i(x)$.

§ 1. Let $k(x)$ be a function (in general, complex-valued) defined for $-\infty < x < \infty$. By $k_\pm(x)$ we shall denote the functions

$$k_\pm(x) = \tfrac{1}{2}(\operatorname{sgn} x \pm 1) k(x).$$

We shall say that $k(x) \in \{\alpha, \beta\}$ if

$$k_+(x) e^{-\xi x} \in L_2(-\infty,\infty) \qquad \text{for } \xi \geqslant \alpha;$$

$$k_-(x)e^{-\xi x} \in L_2(-\infty, \infty) \qquad \text{for } \xi \leqslant \beta.$$

Replacing L_2 by L_1, we define the class $\{\alpha, \beta\}_1$ in the same way. The Fourier transforms of the functions $k_\pm$ will be denoted respectively by $K^\pm$.

Similarly, let $n(x,t)$ be a function defined for $-\infty < x, t < \infty$. Denote by $n_\pm^\pm(x,t)$ the functions

$$n_\pm^\pm(x,t) = \tfrac{1}{4}(\operatorname{sgn} x \pm 1)(\operatorname{sgn} t \pm 1)n(x,t)$$

(the upper signs refer to the first argument, the lower ones to the second).

We shall say that $n(x,t) \in \left\{ \begin{matrix} \alpha, & \beta \\ \gamma, & \delta \end{matrix} \right\}$ if

$$n_+^+(x,t)e^{-\xi x - \tau t} \in L_2 \begin{pmatrix} -\infty, \infty \\ -\infty, \infty \end{pmatrix} \quad \text{for } \xi \geq \alpha, \quad \tau \geq \gamma;$$

$$n_-^-(x,t)e^{-\xi x - \tau t} \in L_2 \begin{pmatrix} -\infty, \infty \\ -\infty, \infty \end{pmatrix} \quad \text{for } \xi \leq \beta, \quad \tau \leq \delta$$

and so on.

§ 2. Let, in equation (1),

$$k_i(x) \in \{\alpha_i, \beta_i\}_1, \quad i = 1, 2; \quad n(x,t) \in \left\{ \begin{matrix} \alpha_3, & \beta_3 \\ \alpha_4, & \beta_4 \end{matrix} \right\}; \quad g(x) \in \{\alpha_5, \beta_5\}, \quad (3)$$

and let the unknown function $f(x) \in \{\alpha, \beta\}$, where $\alpha = \min(\beta_1, \beta_4)$, $\beta = \max(\alpha_2, \alpha_4)$. We note that, generally speaking, this class cannot be enlarged, since the integrals occurring in equation (1) may turn out to be divergent.

Applying to all the functions entering the equation the operations + and −, we write it in the form

$$Tf \equiv v_+(x) + w_1(x) + w_2(x) + u_-(x) = 0, \qquad (4)$$

where the functions v_+, w_1, w_2, and u_- belong to certain classes of the form $\{\xi, \eta\}$ (see (4′)), and moreover $v_+(x) \equiv 0$ for $x < 0$; $u_-(x) \equiv 0$ for $x > 0$. Hence it is not difficult to obtain that the required function $f(x)$ must be such that

$$v_+(x) \in \{\max(\alpha, \beta), \infty\}, \quad u_-(x) \in \{-\infty, \min(\alpha, \beta)\}, \qquad (5)$$

and, for $\alpha < \beta$, must in addition satisfy the conditions

$$v_+(x) + w_2(x) \in \{\alpha, \beta\}, \quad u_-(x) + w_1(x) \in \{\alpha, \beta\}. \tag{6}$$

§ 3. Let $\Phi(z)$ be a function analytic in the half-plane $\operatorname{Im} z > a$ and such that

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^2 \, dx < \infty \quad \text{for } y \geq a.$$

Of such functions we shall say that $\Phi(z) \in \{\{a, \infty\}\}$. In an analogous way we define the classes $\{\{-\infty, b\}\}$ and $\{\{a, b\}\}$ $(a < b)$.

It follows from conditions (5) that the product $v_+(x)e^{-xy}$, $y \geq \max(\alpha, \beta)$, admits a Fourier transform, and the function $V^+(z) \in \{\{\max(\alpha, \beta), \infty\}\}$. In the half-plane $\operatorname{Im} z = y \geq \max(\alpha, \alpha_1, \alpha_3, \alpha_5) = \gamma$ it is constructed by the Fourier transform of each term of the expression $v_+(x)e^{-xy}$. If $\gamma > \max(\alpha, \beta)$, then, by analytic continuation, we define $V^+(z)$ also in the strip $\gamma > \operatorname{Im} z > \max(\alpha, \beta)$. Its boundary value $V^+(\zeta)$, $\zeta = x + i\max(\alpha, \beta)$, as also in the case $\gamma = \max(\alpha, \beta)$, will be called the Fourier transform of the function $v_+(x)e^{-x\max(\alpha,\beta)}$. We proceed similarly in the other cases.

In order to be able to operate separately with each term of $V^+(z)$ and $U^-(z)$, suppose that the functions $k_{i+}(x)$, $i = 1, 2$, and $g_+(x)$ $(k_{i-}(x), g_-(x))$ are such that their Fourier transforms $K_i^+(z)$ and $G^+(z)$ $(K_i^-(z), G^-(z))$, which are analytic functions respectively in the half-planes $\operatorname{Im} z > \alpha_i$ and $\operatorname{Im} z > \alpha_5$ $(\operatorname{Im} z < \beta_i,\ \operatorname{Im} z < \beta_5)$, admit analytic continuation to the line $\zeta = x + i\alpha$ $(\zeta = x + i\beta)$, and under this continuation they may have arbitrary singularities*. We impose analogous restrictions also on the function $n(x, t)$.

---

* In (2–5), when investigating the characteristic equation, only poles are allowed, and at a finite number of points.

§ 4. Let us consider separately three essentially different cases for equation (1).

Case $\alpha = \beta$. Multiplying equation (4) by $e^{-\alpha x}$ and applying the Fourier transform to it, we obtain a certain boundary-value problem for analytic functions (for the characteristic equation—a Riemann problem), which is not difficult to reduce to an equation of the form (2) with respect to the function $F(\zeta)$, $\zeta = x + i\alpha$, if one uses the formulas

$$F^{\pm}(\zeta) = \pm\frac{1}{2}F(\zeta) + \frac{1}{2\pi i}\int_{i\alpha-\infty}^{i\alpha+\infty} \frac{F(\tau)\,d\tau}{\tau - \zeta}, \qquad \zeta = x + i\alpha.$$

The contour Γ is the straight line $\zeta = x + i\alpha$.

Case $\alpha > \beta$. Analogously to how this is done in ([2]), we introduce an auxiliary function $\omega(x)$ and write the equation in the form

$$\omega(x) = v_+(x) + w_1(x) \in \{\alpha, \alpha\}, \qquad -\omega(x) = u_-(x) + w_2(x) \in \{\beta, \beta\}.$$

Multiplying the first of these equalities by $e^{-\alpha x}$, and the second by $e^{-\beta x}$, we transform them by Fourier. The resulting boundary-value problem can be reduced to equation (2) in the same way as in (4). To this end we denote by Γ the contour composed of the line $\zeta = x + i\beta$, traversed from left to right, and the line $\zeta = x + i\alpha$, traversed from right to left, and introduce, as the density of the Cauchy-type integral (over the contour Γ), a new unknown function

$$\Phi(\zeta) = \begin{cases} \Omega(\zeta) - F^+(\zeta), & \zeta = x + i\alpha, \\ \Omega(\zeta) - F^-(\zeta), & \zeta = x + i\beta. \end{cases}$$

Applying the Sokhotski formulas, we express the unknown functions Ω, F^+ and F^- in terms of Φ. Eliminating them from the boundary-value problem, we obtain equation (2).

Case $\alpha < \beta$. Since $Tf \in \{\alpha, \beta\}$ (see (6)), the equations $e^{-\alpha x}Tf = 0$ and $e^{-\beta x}Tf = 0$ can be transformed by Fourier. After the transformation we obtain a certain boundary-value problem for analytic functions with a boundary condition on the lines $\zeta = x + i\alpha$ and $\zeta = x + i\beta$. This problem reduces to a singular equation of the form (2) in the same way as in the case $\alpha > \beta$. The contour Γ will have the former form, only the lines $\zeta = x + i\alpha$ and $\zeta = x + i\beta$ exchange roles. The new unknown function $\Phi(\zeta)$, $\zeta \in \Gamma$, is introduced by the equalities

$$\Phi(\zeta) = F^+(\zeta), \quad \zeta = x + i\alpha; \qquad \Phi(\zeta) = -F^-(\zeta), \quad \zeta = x + i\beta.$$

§ 5. Thus, under the conditions indicated above, equation (1) reduces to a certain singular equation of the form (2), where the contour Γ is a simple closed (at infinity) contour, and the solution must be sought in the class of functions integrable with square modulus*. In the cases $\alpha \geq \beta$, equation (1) will be equivalent to equation (2) with the additional conditions (5), which can also be written in the form

$$V^+(z) \in \{\{\alpha, \infty\}\}, \qquad U^-(z) \in \{\{-\infty, \beta\}\}.$$

If, however, $\alpha < \beta$, then to them one must add also conditions (6) and the conditions

$$\frac{1}{\pi i} \int_{i\alpha-\infty}^{i\alpha+\infty} \frac{\Phi(\tau)\, d\tau}{\tau - \zeta} = \Phi(\zeta), \qquad \zeta = x + i\alpha;$$

$$\frac{1}{\pi i} \int_{i\beta-\infty}^{i\beta+\infty} \frac{\Phi(\tau)\, d\tau}{\tau - \zeta} = -\Phi(\zeta), \qquad \zeta = x + i\beta.$$

* Under certain additional restrictions imposed on the kernels, equation (1) in the cases $\alpha \neq \beta$ can be reduced directly to a Fredholm equation of the second kind.

Equivalence is understood in the sense that all solutions of equation (1) can be obtained from solutions of equation (2) satisfying additional conditions, and conversely—any solution of equation (2) satisfying the additional conditions can be obtained from some solution of equation (1). In the proof one must use the one-to-one invertibility of the Fourier transform and the following fact, which can be obtained by means of Theorems 93 and 95 of the book [6]: functions belonging to the classes $\{\{\alpha, \infty\}\}$ and $\{\{-\infty, \alpha\}\}$ coincide with functions representable by an integral of Cauchy type

$$\frac{1}{2\pi i} \int_{i\alpha-\infty}^{i\alpha+\infty} \frac{\Phi(\tau)\, d\tau}{\tau - z},$$

where the density $\Phi(x + i\alpha) \in L_2(-\infty, \infty)$.

In conclusion we note that the conditions $k_i(x) \in \{\alpha_i, \beta_i\}$, $i = 1, 2$, may be replaced by the conditions $k_i(x) \in \{\alpha_i, \beta_i\}$, but with the additional requirement (besides the restrictions indicated in § 3) that the Fourier transforms of the functions $k_{i+}(x)e^{-\alpha_i x}$ and $k_{i-}(x)e^{-\beta_i x}$ be bounded (see [6], Chs. III and V).

Remark. It is not difficult to see that analogous results can also be obtained for other types of convolution equations ("paired"and integro-differential equations).

I express my deep gratitude to Prof. F. D. Gakhov for a number of valuable suggestions and comments on the present work.

Rostov-on-Don
Pedagogical Institute

Received
26 X 1957

CITED LITERATURE

[1] Yu. I. Cherskii, *Scientific Notes of Kazan State University*, **113**, book 10 (1953).
[2] F. D. Gakhov, Yu. I. Cherskii, ibid., **114**, book 8 (1954).
[3] F. D. Gakhov, Yu. I. Cherskii, DAN, **99**, No. 2 (1954).
[4] Yu. I. Cherskii, Dissertation, Tbilisi, 1956.
[5] F. D. Gakhov, Yu. I. Cherskii, *Izv. AN SSSR, Ser. Mat.*, **20**, No. 1 (1956).
[6] E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 1948.

Note: Figure translations are in progress. See original paper for figures.