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Abstract

Full Text

THEORY OF ELASTICITY

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STRESSES IN A SYMMETRICALLY HEATED SPHERICAL SHELL WHOSE MECHANICAL PROPERTIES DEPEND ON TIME AND TEMPERATURE

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1. Let a closed spherical shell be given, with external and internal radii a_1 and a_2, free of external forces and initial stresses, at a uniformly distributed temperature. Suppose that, beginning at some moment, the temperature in the shell becomes nonuniformly distributed, and assume that the temperature $T = T(r, t)$ is known. Then stresses and strains of a definite magnitude will arise in it.

We determine the radial displacement $u(r, t)$ and the stresses arising in the body in question in the case where the Lamé coefficients λ and μ and the relaxation characteristics depend on temperature, and also taking into account aging of the material, i.e., the variability of λ and μ with time t.

Generalizing in the appropriate way the Volterra dependences ([1]), we write the stress components in spherical coordinates

$$\sigma_r = \lambda\theta + 2\mu\frac{\partial u}{\partial r} - \int_{t_0}^{t}\left[\varphi(t,\tau;r)\,\theta + 2\psi(t,\tau;r)\frac{\partial u}{\partial r}\right]d\tau - \alpha\beta T,$$

$$\sigma_\theta = \sigma_\varphi = \lambda\theta + 2\mu\frac{u}{r} - \int_{t_0}^{t}\left[\varphi(t,\tau;r)\,\theta + 2\psi(t,\tau;r)\frac{u}{r}\right]d\tau - \alpha\beta T, \qquad (1)$$

where $\theta(r, t)$ is the volume expansion; $\varphi(t, \tau; r)$ and $\psi(t, \tau; r)$ are the relaxation kernels; λ, μ, and $\beta = 3\lambda + 2\mu$, as well as the coefficient of linear expansion α, depend on the coordinate r and time t, since, by assumption, the temperature of the shell is considered a known function of r and t.

Substituting σ_r, $\sigma_\theta = \sigma_\varphi$ from (1) into the usual equilibrium equation, we obtain the initial integro-differential equation, which we present in the form

$$\frac{\partial}{\partial r}\left[(\lambda+2\mu)\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u)\right]-\frac{4}{r}\frac{\partial\mu}{\partial r}u-\frac{\partial}{\partial r}(\alpha\beta T)+\frac{4}{r}\int_{t_0}^{t}\frac{\partial\psi(t,\tau,r)}{\partial r}u(r,\tau)\,d\tau=$$

$$=\int_{t_0}^{t}\frac{\partial}{\partial r}\left\{[\varphi_1(t,\tau;r)+2\psi(t,\tau;r)]\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u)\right\}d\tau. \tag{2}$$

Put

$$\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u)\right]=z(r,t). \tag{3}$$

From (3) it follows that

$$u(r,t)=\frac{1}{3r^2}\int_{a_1}^{r}(r^3-\rho^3)z(\rho,t)\,d\rho+c_1(t)r+r^{-2}c_2(t). \tag{4}$$

where c_1 and c_2 are arbitrary functions of time, which are determined from the boundary conditions

$$\left\{\lambda\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)+2\mu\frac{\partial u}{\partial r}-\alpha\beta T-\int_{t_0}^{t}\left[\varphi(t,\tau;r)\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)+2\psi(t,\tau;r)\frac{\partial u}{\partial r}\right]d\tau\right\}_{r=a_k}=0 \tag{5}$$

$(k=1,2)$ after $z(r,t)$ has been found.

Substituting (3) and (4) into equation (2), after the corresponding transformations we obtain the mixed two-dimensional integral equation

$$z(r,t)+\int_{a_1}^{r}K_1(t;r,\rho)z(\rho,t)\,d\rho-\int_{t_0}^{t}K_2(t,\tau;r)z(r,\tau)\,d\tau=$$

$$=\int_{t_0}^{t}\int_{a_1}^{r}K_3(t,\tau;r,\rho)z(\rho,\tau)\,d\rho\,d\tau+f(r,t,c_1,c_2), \tag{6}$$

where

$$K_1(t;r,\rho)=\frac{1}{\lambda+2\mu}\left[\frac{\partial}{\partial r}(\lambda+2\mu)-\frac{4(r^3-\rho^3)}{3r^3}\frac{\partial\mu}{\partial r}\right],$$

$$K_2(t,\tau;r)=\frac{\varphi(t,\tau;r)+2\psi(t,\tau;r)}{\lambda+2\mu},$$

$$K_3(t,\tau;r,\rho) = \frac{1}{\lambda+2\mu}\left\{\frac{\partial}{\partial r}\left[\varphi(t,\tau;r)+2\psi(t,\tau;r)\right] - \frac{4(r^3-\rho^3)}{3r^3}\frac{\partial\psi(t,\tau;r)}{\partial r}\right\},$$

$$f(r,t;c_1,c_2) = \frac{1}{\lambda+2\mu}\left\{\frac{4}{r}\left(c_1 r+\frac{c_2}{r^2}\right)\frac{\partial\mu}{\partial r} - \frac{4}{r}\int_{t_0}^{t}\left(c_1 r+\frac{c_2}{r^2}\right)\frac{\partial\psi(t,\tau;r)}{\partial r}\,d\tau\right.$$

$$\left.-3c_1\frac{\partial}{\partial r}(\lambda+2\mu) + 3\int_{t_0}^{t} c_1(\tau)\frac{\partial}{\partial r}\left[\varphi(t,\tau;r)+2\psi(t,\tau;r)\right]\,d\tau\right\}.$$

Determining z from equation (6) and then substituting into (4), after the corresponding transformations we obtain the formula for the displacement

$$u = c_1 A_1(t,r) - \int_{t_0}^{t}\left[A_2(t,\tau;r)c_1(\tau) - A_3(t,\tau;r)c_2(\tau)\right]\,d\tau + c_2 r^{-2}, \qquad (7)$$

where

$$A_1(t,r) = r + \frac{1}{3r^2}\int_{a_1}^{r}\frac{D(t;r,\rho)}{\lambda+2\mu}\left[4\frac{\partial\mu}{\partial\rho} - 3\frac{\partial}{\partial\rho}(\lambda+2\mu)\right]\,d\rho,$$

$$D(t;r,\rho) = r^3-\rho^3 - \int_{\rho}^{r}(r^3-\rho_1^3)R_1(t;\rho_1,\rho)\,d\rho_1,$$

$$A_2(t,\tau;r) = \frac{1}{3r^2}\int_{a_1}^{r}\frac{D(t;r,\rho)}{\lambda+2\mu}\left[4\frac{\partial\psi(t,\tau;r)}{\partial\rho} - 3\frac{\partial}{\partial\rho}(\varphi(t,\tau;\rho)+2\psi(t,\tau;\rho))\right]\,d\rho,$$

$$A_3(t,\tau;r) = \frac{4}{3r^2}\int_{a_1}^{r}\frac{\rho^{-3}}{\lambda+2\mu}\left[E(t,\tau;r,\rho)\frac{\partial\mu}{\partial\rho} - \int_{\tau}^{t}E(t,\tau_1;r,\rho)\frac{\partial\psi(\tau_1,\tau;\rho)}{\partial\rho}\,d\tau_1\right],$$

$$E(t,\tau;r,\rho) = (r^3-\rho^3)R_2(t,\tau;\rho) + \int_{\rho}^{r}(r^3-\rho_1^3)R_3(t,\tau;\rho_1,\rho)\,d\rho_1,$$

$$R_3(t,\tau;r,\rho) = R(t,\tau;r,\rho) + \int_{\tau}^{t}R_2(t,\tau_1;r)\,R(\tau_1,\tau;r,\rho)\,d\tau_1 -$$

$$-\int_{\rho}^{r}R_1(t;r,\rho_1)\,R(\tau_1,\tau;\rho_1,\rho)\,d\rho_1 -$$

$$-\int_{\tau_1}^{t} d\tau_1 \int_{\rho}^{r} R_1(\tau_1; r, \rho_1)\, R_2(t, \tau_1; \rho_1)\, R(\tau_1, \tau; \rho_1, \rho)\, d\rho_1;$$

$R_1(t; r, \rho)$ and $R_2(t, \tau; r)$ are the resolvents of the kernels $K_1(t; r, \rho)$ and $K_2(t, \tau; r)$, respectively; $R(t, \tau; r, \rho)$ is the resolvent of the kernel

$$\left[Q(t, \tau; r, \rho) + \int_{\tau}^{t} Q(t, \tau_1; r, \rho)\, R_2(\tau_1, \tau; \rho)\, d\tau_1 \right];$$

$$Q(t, \tau; r, \rho) = K_3(t, \tau; r, \rho) - K_2(t, \tau; r) R_1(\tau; r, \rho) - \int_{\rho}^{r} K_3(t, \tau; r, \rho_1) R_1(\tau; \rho_1, \rho)\, d\rho_1.$$

The functions $c_1(t)$ and $c_2(t)$ appearing in (7) are determined from the system of ordinary integral equations

$$\sum_{i=1}^{2} \left[\alpha_{ik} c_k(t) - \int_{t_0}^{t} \Phi_{ik}(t, \tau) c_k(\tau)\, d\tau \right] = F_k, \qquad (8)$$

where

$$\alpha_{ik} = \left\{ \frac{1 - (-1)^i}{2} \left[(\lambda + 2\mu) \frac{\partial A_1}{\partial r} + 2\lambda \frac{A}{r} \right] - 2\mu \frac{(-1)^i + 1}{r^3} \right\}_{r = a_k} \qquad (i, k = 1, 2);$$

$$F_k = \alpha_k \beta_{kT} k; \qquad \Phi_{ik}(t, \tau) = \left\{ \frac{1 - (-1)^i}{2} (\varphi + 2\psi) \frac{\partial A_1}{\partial r} - 2\psi \frac{(-1)^i + 1}{r^3} + \right.$$

$$\left. + \frac{(-1)^k + 1}{2} \left[(\lambda + 2\mu) \frac{\partial A_{i+1}}{\partial r} - \frac{2\lambda}{r} A_{i+1} - \int_{\tau}^{t} \left((\varphi + 2\psi)_{\tau_1} \frac{\partial A_{i+1}}{\partial r} + \frac{2\varphi}{r} A_{i+1} \right) d\tau_1 \right] \right\}_{r = a_k}.$$

System (8) has a unique solution.
From the first equation of system (8) it follows that

$$c_1(t) = \frac{F_1}{a_{11}} + \int_{t_0}^{t} \frac{H(t, \tau)}{a_{11}(\tau)} F_1(\tau)\, d\tau - \frac{a_{21}}{a_{11}} c_2(t) + \int_{t_0}^{t} H_1(t, \tau) c_2(\tau)\, d\tau, \qquad (9)$$

where

$$H_1(t,\tau) = \alpha_{11}^{-1}(\tau)\left[\Phi_{21}(t,\tau) - \alpha_{21}H(t,\tau) + \int_\tau^t H(t,s)\Phi_{21}(s,\tau)\,ds\right],$$

$H(t,\tau)$ is the resolvent of the kernel $\Phi_{11}(t,\tau)/\alpha_{11}$.

Substituting (9) into the second equation of system (8), we obtain an integral equation with one unknown c_2. Determining c_2 from the equation thus obtained and then again using (9), we obtain formulas determining c_k.

Knowing the displacement u, it is easy to determine $\sigma_r,\ \sigma_\theta = \sigma_\varphi$.

2. The obtained results are considerably simplified if it is assumed that $\lambda = \mu = \lambda_0 e^{-mT}$, $\varphi(t,\tau;r) = \psi(t,\tau;r) = \varphi_0(t,\tau)e^{-mT}$, where m is a constant for the given material; λ_0 and $\varphi_0(t,\tau)$ are the Lamé coefficient and the relaxation kernel at $T = 0$; the temperature T still depends on r and t.

The decrease of the physico-mechanical characteristics is proportional to many-

···for a viscosity that decreases exponentially with increasing temperature, is confirmed experimentally ([2]).

The substitution (3) leads to an ordinary integral equation with respect to z, with a fairly simple kernel.

As a result we shall have

$$u(r,t) = \int_{a_1}^r N_1(t;r,\rho)\,T_*(\rho,t)\,d\rho + c_1\left[r + \frac{5m}{3}\int_a^r N_1(t;r,\rho)\frac{\partial T}{\partial\rho}\,d\rho\right] + \\ + c_2\left[\frac{1}{r^2} - \frac{4m}{3}\int_{a_1}^r N_1(t;r,\rho)\frac{1}{\rho^3}\frac{\partial T}{\partial\rho}\,d\rho\right], \qquad T_* = T_1 + \int_{t_0}^t P(t,\tau)T_1\,d\tau, \tag{10}$$

$$N_1(t;r,\rho) = \frac{1}{3r^2}\left[r^3 - \rho^3 + \int_\rho^r (r^3-\rho_1^3)N(t;\rho_1,\rho)\,d\rho_1\right];$$

$P(t,\tau)$ and $N(t;r,\rho)$ are the resolvents of the kernels $\dfrac{\varphi_0(t,\tau)}{\lambda_0}$ and

$$\frac{m}{9}\left(5 + \frac{4\rho^3}{r^3}\right)\frac{\partial T}{\partial r},$$

respectively;

$$T_1 = 5\left[\frac{\partial}{\partial r}(\alpha T) - m\alpha T\frac{\partial T}{\partial r}\right].$$

The functions $c_1(t)$ and $c_2(t)$ appearing in (10) are expressed by the formulas

$$c_1 = \frac{a_1^3 B_2 \Phi_{a_1} - a_2^3 \Phi_{a_2}}{a_2^3 B_1 - a_1^2 B_2}, \qquad c_2 = \frac{5a_1^3 a_2^3 (\Phi_{a_2} - B_1 \Phi_{a_1})}{4(a_2^3 B_1 - a_1^2 B_2)},$$

$$B_1 = 1 + \frac{m}{3} \int_{a_1}^{a_2} \frac{\partial T}{\partial \rho} \left[\frac{\partial N_1(t; r, \rho)}{\partial r} + \frac{N_1(t; r, \rho)}{a_2} \right]_{r=a_2} d\rho, \tag{11}$$

$$B_2 = 1 + m a_2^3 \int_{a_1}^{a_2} \frac{1}{\rho^3} \frac{\partial T}{\partial \rho} \left[\frac{\partial N_1(t; r, \rho)}{\partial r} + \frac{2N_1(t; r, \rho)}{3a_2} \right]_{r=a_2} d\rho;$$

Φ_{a_1} and Φ_{a_2} are the values of the function

$$\Phi(r, t) = \alpha T + \int_{t_0}^{t} P(t, \tau) \alpha T \, d\tau$$

at $r = a_1$ and $r = a_2$, respectively.

Formulas (11) are obtained as the result of solving the system of algebraic equations formed by using the boundary conditions (5).

As an illustration, let us consider a steel hollow ball with the following parameters: $m = 0.0008$ deg^{-1} (according to the experimental data of Zaikov ([2])), $a_1 = 30$ cm, $a_2 = 33$ cm, $T = 9900 r^{-1} + 670$, $T(a_1) = 1000^\circ$, $T(a_2) = 970^\circ$, $\alpha = 11 \cdot 10^{-6}$ deg^{-1},

$$P(s) = 0.0175 (s/33)^{0.75} \exp\left(-\sqrt[4]{s/33}\right)$$

(according to ([3])). Approximately we obtain

$$u = (0.1165 - 0.00762 \exp(-\sqrt[4]{t/33}))(0.9853 - 0.04097 r - 17.85 r^{-1} - 525.5 r^{-2} - 2253 r^{-3} + 1898 r^{-4} + 46 r^{-3} \ln r + 3.53$$

In particular, for example, the steady-state value of the strain at the outer surface will be

$$\varepsilon = 0.00507.$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.