HYDROMECHANICS

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Abstract

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THE ZEMPLEN THEOREM IN MAGNETO-HYDRODYNAMICS

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A number of works have been devoted to discontinuities of various kinds in a medium with infinite conductivity and negligibly small dissipation in the presence of a magnetic field (see the review (^{1})). Of special interest are discontinuities of the shock-wave type, in which matter passes through the surface of discontinuity and a change in the thermodynamic quantities occurs. In the book (^{2}) it is shown that, for small magnitudes of such discontinuities, under the same conditions as for ordinary shock waves, only compression waves are possible. We shall prove this assertion for arbitrary magnitudes of the discontinuities.

Discontinuities of the shock-wave type are described by the equations (^{2}):

$$j^2(V_1H_1 - V_2H_2) = \frac{H_n^2}{4\pi}(H_1 - H_2),$$

$$j^2 = \frac{8\pi(p_2 - p_1) + H_2^2 - H_1^2}{8\pi(V_1 - V_2)}, \tag{1}$$

$$\varepsilon(p_2, V_2) - \varepsilon(p_1, V_1) + (V_2 - V_1)\frac{p_2 + p_1}{2} + \frac{1}{16\pi}(H_2 - H_1)^2(V_2 - V_1) = 0.$$

Here p, V, H, j are, respectively, the pressure, specific volume, tangential magnetic field, and flux of matter through the discontinuity; H_n is the normal component of the magnetic field; $\varepsilon(p, V)$ is the internal energy per unit mass of matter. Index 1 refers to quantities before the jump, index 2 to quantities after the jump.

We shall consider only substances for which

$$\left(\frac{\partial^2 p}{\partial V^2}\right)_S > 0, \qquad \left(\frac{\partial p}{\partial S}\right)_V > 0.$$

Equations (1) determine all possible types of shock waves in magnetohydrodynamics.

In order to obtain the curves $p_2(V_2)$ of the type of Hugoniot curves, it is necessary to eliminate H_2 from the last equation with the aid of the other two equations. Eliminating j^2, we obtain

$$\left(p_2 - p_1 + \frac{H_2^2 - H_1^2}{8\pi}\right)(V_1 H_1 - V_2 H_2) = \frac{H_n^2}{4\pi}(H_1 - H_2)(V_1 - V_2). \qquad (2)$$

In what follows we shall assume that $H_n \neq 0$, $H_1 \neq 0$. There exist three solutions of equation (2), and it can be shown that for weak discontinuities these solutions are real and give three different Hugoniot curves $p_2(V_2)$ passing through the point p_1, V_1.

It can be shown, however, that for $H_1 \neq 0$ equation (2) has only one real root for sufficiently large p_2. This can be verified by seeking the solution of (2) by means of two different iteration—

processes:

$$H_{2k+1} = \frac{1}{V_2}\left[V_1 H_1 - \frac{2H_n^2(H_1 - H_{2k})(V_1 - V_2)}{8\pi(p_2 - p_1) + H_{2k}^2 - H_1^2}\right]; \qquad H_{20} = \frac{V_1}{V_2} H_1; \qquad (\mathrm{I})$$

$$H_{2k+1} = \pm\sqrt{-8\pi(p_2 - p_1) + H_1^2 + \frac{2H_n^2(H_1 - H_{2k})(V_1 - V_2)}{V_1 H_1 - V_2 H_{2k}}}; \qquad (\mathrm{II})$$

$$H_{20} = \pm\sqrt{-8\pi p_2}.$$

It can be shown that these processes converge for sufficiently large p_2. (We note that $V_1 H_1$ cannot be neglected in comparison with $V_2 H_2$, since this leads to incorrect conclusions near $V_2 = 0$.)

According to (2), the entropy increases near $p_2 = p_1$ with increasing p_2 along each of these three curves. We shall show that along any of these Hugoniot curves the entropy S_2 increases monotonically and that always $p_2 > p_1$. The proof is analogous to the well-known proof in gas dynamics (^{3}).

It is convenient to introduce the auxiliary quantities:

$$p_2^* = p_2 + \frac{H_2^2}{8\pi}, \qquad p_1^* = p_1 + \frac{H_1^2}{8\pi}.$$

To any Hugoniot curve $p_2(V_2)$ there corresponds some curve $p_2^*(V_2)$, according to the chosen branch $H_2(p_2, V_2)$. The lines $j^2 = \mathrm{const}$ in the plane p_2^*, V_2 will simply be straight lines issuing from the point p_1^*, V_1.

Consider the Hugoniot function

$$\mathcal{H}_+(p_2, V_2) = \varepsilon(p_2, V_2) - \varepsilon(p_1, V_1) + \frac{p_2 + p_1}{2}(V_2 - V_1) +$$

$$+ \frac{1}{16\pi}(H_{2+} - H_1)^2 (V_2 - V_1).$$

Here the plus sign indicates the fact that we have chosen a definite branch $H_+(p_2, V_2)$ as a solution of equation (2). Computing the differential $\mathcal{H}_+$, we obtain:

$$d\mathcal{H}_+ = T_2\, dS_2 + \left[\frac{p_1^* - p_2^*}{2}\, dV_2 + \frac{1}{2}(V_2 - V_1)\, dp_{2+}^*\right] +$$

$$+ \frac{1}{8\pi}(H_{2+}^2 - H_1 H_{2+})\, dV_2 - \frac{1}{8\pi} H_1 (V_2 - V_1)\, dH_{2+}. \tag{3}$$

From equations (1) we have the equalities:

$$j_+^2 = \frac{p_{2+}^* - p_1^*}{V_1 - V_2}; \qquad H_{2+} = H_1 \frac{H_n^2 - 4\pi j_+^2 V_1}{H_n^2 - 4\pi j_+^2 V_2}. \tag{4}$$

Computing dj_+^2 and dH_{2+} and substituting in (3), we obtain:

$$d\mathcal{H}_+ = T_2 dS_2 + \left[1 + \frac{H_1^2 H_n^2}{(H_n^2 - j_+^2 V_2)^2}\right] \left(\frac{p_1^* - p_{2+}^*}{2}\, dV_2 - \frac{V_2 - V_1}{3}\, dp_{2+}^*\right). \tag{5}$$

Hence we conclude that along the Hugoniot curve dS_2 does not change sign, provided only that the corresponding curve $p_{2+}^*(V_2)$ nowhere touches the ray $j^2 = \text{const}$.

Along the curve $j_+^2(p_2, V_2) = \text{const}$, $d\mathcal{H}_+ = T_2 dS_2$, therefore S_2 and $\mathcal{H}_+$ have extrema at the same points. In addition, along this same curve

$$\frac{dp_{2+}^*}{dV_2} = -j_+^2 = \left(\frac{\partial p_2}{\partial V_2}\right)_S + \left(\frac{\partial p_2}{\partial S_2}\right)_V \frac{dS_2}{dV_2} + \frac{1}{8\pi}\left(\frac{\partial H_{2+}^2}{\partial V_2}\right)_{j^2}.$$

At the point where $dS_2/dV_2 = 0$, using (4), we obtain:

$$\frac{d^2 p_2^*}{dV_2^2} = \left(\frac{\partial^2 p_2}{\partial V_2^2}\right)_S + \left(\frac{\partial p_2}{\partial S_2}\right)_V \frac{d^2 S_2}{dV_2^2} + \frac{12\pi H_2^2 + j_+^4}{\left(H_2^2 - 4\pi j_+^2 V_2\right)^2} = 0.$$

Using our assumptions about the function $p(V, S)$, we conclude that the stationary point of S_2 on the ray $j_+^2 = \text{const}$ is a maximum. Since at the intersection points of the ray j_+^2 = const with the Hugoniot curve $\mathcal{H}_+ = 0$, $\mathcal{H}_+$ has an extremum between these points, it follows that S_2 has a maximum at this same point. Therefore there is no more than one intersection point (not counting the point p_1, V_1), and $dS_2/dV_2 > 0$ at any such point. This proves the impossibility of tangency of the Hugoniot curve by a ray.

Since for small discontinuities $dS_2 > 0$, $dj_+^2 > 0$, it follows from this that always $dS_2 > 0$, $dj_+^2 > 0$ along any Hugoniot curve. Hence it follows that $p_2 > p_1$ at any discontinuity (since $(\partial p/\partial S)_V > 0$).

Since we have shown that for very large p_2 the two roots $H_2(p_2, V_2)$ become imaginary, two of the Hugoniot curves must terminate at a finite p_2 (under the assumption $H_1 \neq 0$).

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2. L. D. Landau, E. M. Lifshitz, *Electrodynamics of Continuous Media*, Moscow, 1957.
3. R. Courant, K. Friedrichs, *Supersonic Flow and Shock Waves*, IL, 1950.

Note: Figure translations are in progress. See original paper for figures.

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