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Abstract

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ASYMPTOTIC SOLUTIONS OF THE EQUATION OF MOTION OF A NONLINEAR OSCILLATORY SYSTEM WITH ONE DEGREE OF FREEDOM AND SLOWLY VARYING PARAMETERS

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In the present work we consider the equation

$$\frac{d^2y}{dt^2} + \varepsilon f(\tau, y)\frac{dy}{dt} + F(\tau, y) = 0, \tag{1}$$

where ε is a small parameter, $\tau = \varepsilon t$ is slow time. This equation may be interpreted as the equation of motion of a material point of mass equal to unity under the action of the principal force $-F(\tau, y)$ and a small dissipative force $-\varepsilon f(\tau, y)\dfrac{dy}{dt}$. The aim of the work is to compute expressions suitable for investigating the solution of equation (1) with an error of order ε for $0 \le t \le \tau_0/\varepsilon$. It is assumed that the solution of equation (1) has an oscillatory character.

To solve this problem, the method of "standard" equations is used in the work ([1,2]). The essence of this method is that the solution of the given equation is expressed in terms of the solution of some analogous equation (the "standard" equation), which possesses all the features of the given equation. In the case under consideration, the equation chosen as the "standard" one is

$$\varphi^2(\tau)\frac{\partial^2 y_0}{\partial\omega^2} + F(\tau, y_0) = 0, \tag{2}$$

where $\varphi(\tau)$ is an arbitrary function. In this equation the variables τ and ω are regarded as independent. It is assumed that this equation has a family of solutions periodic in ω. Its general solution depends on three arbitrary functions $\lambda(\tau)$, $\Omega(\tau)$, and $\varphi(\tau)$. One of these functions is additive to the argument ω; for definiteness we shall suppose that this function is $\Omega(\tau)$.

From the general theorems on the dependence of solutions of differential equations on a parameter it follows that if, in the expression for $y_0(\tau, \omega)$, one expresses τ and ω in terms of t by means of the relations

$$\frac{d\tau}{dt} = \varepsilon, \qquad \frac{d\omega}{dt} = \varphi(\tau), \tag{3}$$

then the function of t obtained in this way will differ from the solution of equation (1) by quantities of order ε on each time interval of order unity. In order that this function be suitable for investigating the solution of equation (1) for $0 \le t \le \tau_0/\varepsilon$, it is necessary to determine the arbitrary functions $\lambda(\tau)$, $\Omega(\tau)$, and $\varphi(\tau)$ in an appropriate manner. To this end we introduce the function $\tilde{y}(t)$

$$\tilde{y}(t) = y_0(\tau, \omega) + \varepsilon y_1(\tau, \omega). \tag{4}$$

Here τ and ω are expressed in terms of t by means of relations (3). We note that when $0 \le t \le \tau_0/\varepsilon$, then $0 \le \tau \le \tau_0$ and

$$0 \le \omega \le \frac{\tau_0 \max \varphi(\tau)}{\varepsilon} < \infty.$$

In what follows we shall assume that the functions $f(\tau, y)$ and $F(\tau, y)$ are defined and have a sufficient number of bounded derivatives with respect to τ and y for $0 \le \tau \le \tau_0$ and $0 \le |y| \le w$ (w is some constant). Substituting expression (4) into equation (1) and using (3), we obtain

$$\varphi^2(\tau)\frac{\partial^2 y_0}{\partial \omega^2} + F(\tau, y_0) + \varepsilon \left\{ \varphi^2(\tau)\frac{\partial^2 y_1}{\partial \omega^2} + F_y(\tau, y_0)y_1 + 2\varphi(\tau)\frac{\partial^2 y_0}{\partial \omega \partial \tau} + \right.$$
$$\left. + \left[\varphi'(\tau) + f(\tau, y_0)\varphi(\tau)\right]\frac{\partial y_0}{\partial \omega} \right\} + \varepsilon^2 \Delta\left(\tau, \frac{\partial^{k+l} y_0}{\partial \tau^k \partial \omega^l}, \frac{\partial^{k+l} y_1}{\partial \tau^k \partial \omega^l}, \varepsilon\right) = 0 \tag{5}$$

$$(k, l = 0, 1, 2).$$

We note that the coefficient of ε^2 (the expression for it is easy to write down) is, for $0 \le \varepsilon \le \varepsilon_0$, a function bounded for bounded values of its arguments*.

The terms of order unity in this equation cancel completely by virtue of the choice of the "reference" equation. The terms of order ε shall be compensated by means of the choice of the function y_1. For this purpose it is evidently necessary to determine it from the equation

$$\varphi^2(\tau)\frac{\partial^2 y_1}{\partial \omega^2} + F_y(\tau, y_0)y_1 = -2\varphi(\tau)\frac{\partial^2 y_0}{\partial \omega \partial \tau} - \left[\varphi'(\tau) + f(\tau, y_0)\varphi(\tau)\right]\frac{\partial y_0}{\partial \omega}. \tag{6}$$

In this equation, just as in (2), the variables τ and ω are independent.

Since the function $\partial y_0/\partial\omega$ is a particular solution of the homogeneous equation in (6), the expression for the function y_1 is easy to write down. With regard to the terms of order ε^2, we shall require that the coefficient of ε^2 be bounded for $0 \le t \le \tau_0/\varepsilon$. In view of the remarks made above, for this it is sufficient that the functions $y_0(\tau,\omega)$ and $y_1(\tau,\omega)$, together with their derivatives up to and including the second order, be bounded for $0 \le \tau \le \tau_0$ and $0 \le \omega < \infty$. The latter will hold under the conditions: a) $\varphi(\tau) \ge \delta > 0$; b) $y_i(\tau,\omega) = y_i(\tau,\omega+T_\omega)$ $(i = 1, 2)$, where T_ω does not depend on τ.

Indeed, under condition a), from the theorem on the existence and differentiability of solutions of differential equations it follows that the solutions of equations (2) and (6), together with their first and second derivatives with respect to τ and ω, are bounded for $0 \le \tau \le \tau_0$ and $0 \le \omega \le T_\omega$, provided only that the function $\varphi(\tau)$ is twice differentiable. The differentiability of the function $\varphi(\tau)$ follows from relation (7), which is given below. Since the period of the functions $y_i(\tau,\omega)$ does not depend on τ, all their derivatives with respect to τ and ω are periodic in ω with the same period. Therefore, from the boundedness of the functions $y_i(\tau,\omega)$ and their derivatives for $0 \le \tau \le \tau_0$ and $0 \le \omega \le T_\omega$, there follows their boundedness for $0 \le \tau \le \tau_0$ and $0 \le \omega < \infty$.

Condition b) makes it possible to formulate conditions for determining the arbitrary functions $\lambda(\tau)$, $\Omega(\tau)$, and $\varphi(\tau)$. The condition for determining the function $\lambda(\tau)$ is the assurance of the independence, with respect to τ, of the period in ω of the function $y_0(\tau,\omega)$. From this condition the function $\lambda(\tau)$ can easily be determined in each concrete case. The conditions for determining the functions $\Omega(\tau)$ and $\varphi(\tau)$ ensure the periodicity of $y_1(\tau,\omega)$ in ω and are obtained as a result of investigating the expression for this function. These conditions have the form

$$\Omega(\tau) = 0; \qquad \frac{d}{d\tau}\left[\varphi(\tau)\int_{a_1}^{a_2}\left(\frac{\partial y_0}{\partial\omega}\right)^2 d\omega\right] + \varphi(\tau)\int_{a_1}^{a_2} f(\tau,y_0)\left(\frac{\partial y_0}{\partial\omega}\right)^2 d\omega = 0. \tag{7}$$

Here a_1 and a_2 are successive zeros of the derivative $\partial y_0/\partial\omega$.

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* In the present work, whenever the boundedness of some function is mentioned, it is understood that this function is bounded by a constant independent of ε.

With the aid of the indicated conditions, the arbitrary functions $\lambda(\tau)$, $\Omega(\tau)$, and $\varphi(\tau)$ are determined uniquely. Of these conditions, the most essential is the second of conditions (7). It is a differential relation. In the case when $f(\tau,y) \equiv f(\tau)$, it can be integrated and may be written in the form

$$\varphi(\tau)\int_{a_1}^{a_2}\left(\frac{\partial y_0}{\partial\omega}\right)^2 d\omega = D\exp\left[-\int_{\tau_0}^{\tau} f(\tau)\,d\tau\right]. \qquad (8)$$

Here D is an arbitrary constant.

Let us summarize the above in the form of the following theorem.

Theorem. *If, for $0 \le \tau \le \tau_0$ and $0 \le |y| \le w$, the functions $f(\tau, y)$ and $F(\tau, y)$ are sufficiently smooth; the "reference" equation for $0 \le \tau \le \tau_0$ has a solution periodic in ω, with a period independent of τ; the arbitrary functions $\Omega(\tau)$ and $\varphi(\tau)$ are determined from conditions (7), and in such a way that for $0 \le \tau \le \tau_0$, $\varphi(\tau) \ge \delta > 0$, then the function*

$$\tilde{y}(t) = y_0\left[\varepsilon t, \int \varphi(\varepsilon t)\,dt\right] + \varepsilon y_1\left[\varepsilon t, \int \varphi(\varepsilon t)\,dt\right] \qquad (9)$$

satisfies equation (1) for $0 \le t \le \tau_0/\varepsilon$ and $0 \le \varepsilon \le \varepsilon_0$, up to terms of order ε^2.

Since the term of order ε in (9) is a small oscillatory correction to the principal term, it can usually be neglected. Therefore the asymptotic formulas for the solution of equations (1) and its derivative have the form

$$y_0(t) = y_0(\tau, \omega); \qquad \left(\frac{dy}{dt}\right)_0 = \varphi(\tau)\frac{\partial y_0}{\partial\omega}, \qquad (10)$$

where $\tau = \varepsilon t$; $\omega = \omega_0 + \int_{t_0}^{t} \varphi(\varepsilon t)\,dt$; ω_0 is an arbitrary constant. From the equality $y_0(\tau, \omega) = y_0(\tau, \omega + T_\omega)$ it follows that the instantaneous period of oscillation $T(\tau)$ is connected with the function $\varphi(\tau)$ by the formula $T(\tau) = T_\omega/\varphi(\tau)$, so that the function $\varphi(\tau)$ is the instantaneous frequency of the oscillations.

With the aid of the asymptotic formulas (10), it is not difficult to give a physical interpretation of condition (7) for determining the function $\varphi(\tau)$. It turns out that the first term in (7) is the rate of change of the product of the instantaneous period of oscillation by the mean-over-a-period value of the kinetic energy of the system whose motion is described by equation (1), while the second term is the product of the instantaneous period by the mean value of the dissipative function. In the case when $f(\tau, y) = 0$, the product of the instantaneous period by the mean value of the kinetic energy is constant. This result is known in the theory of adiabatic invariants ([3]).

Let us next dwell briefly on the equation

$$\frac{d^2y}{dt^2} + \varepsilon f(\tau, y)\frac{dy}{dt} + a_0(\tau) + a_1(\tau)y + a_2(\tau)y^2 + a_3(\tau)y^3 = 0. \qquad (11)$$

Such an equation is often encountered in applications. The asymptotic solutions in this case are expressed in terms of Jacobi elliptic functions. The solution of the "reference" equation should be sought in one of the forms

$$y_0(\tau,\omega)=\frac{\alpha(\tau)\sigma(\tau,\omega)+\beta(\tau)}{\gamma(\tau)\sigma(\tau,\omega)+1},\qquad y_0(\tau,\omega)=\frac{\alpha(\tau)\sigma^2(\tau,\omega)+\beta(\tau)}{\gamma(\tau)\sigma^2(\tau,\omega)+1}. \tag{12}$$

Here by $\sigma(\tau,\omega)$ is meant any one of the functions $\mathrm{sn}[K(\nu(\tau))\omega,\nu(\tau)]$; $\mathrm{cn}[K(\nu(\tau))\omega,\nu(\tau)]$; $\mathrm{dn}[K(\nu(\tau))\omega,\nu(\tau)]$ ($K(\nu)$ is the complete elliptic inte-

integral of the first kind; ν is the square of the modulus). The function $\lambda(\tau)$, which in the present case is the coefficient of ω, is chosen so that, with respect to ω, the functions (12) have a period independent of τ, namely $T_\omega = 4$. The relations for determining the functions $\alpha(\tau)$, $\beta(\tau)$, $\gamma(\tau)$, and $\nu(\tau)$ are obtained by substituting expressions (12) into the "reference" equation and equating to zero the coefficients of the different powers of the function $\sigma(\tau,\omega)$. The relation for determining the instantaneous frequency is obtained from equality (7). Particularly simple formulas are obtained in the cases when either $a_0(\tau)\equiv a_2(\tau)\equiv 0$, or $a_3(\tau)\equiv 0$. In the first case the solution of the "reference" equation should be sought in the first of the forms (12) with $\gamma(\tau)\equiv\beta(\tau)\equiv 0$, and in the second case—in the second of the forms (12) with $\gamma(\tau)\equiv 0$.

In those cases where the solution of the "reference" equation cannot be expressed in terms of special functions, it can be sought by means of the Bubnov-Galerkin method.

The study of the asymptotic behavior of the solution of equations (1) was also the subject of papers ([4–6]). From the point of view of the formulation of the problem, paper ([4]) is close to the present paper. In paper ([4]) the problem of directly computing asymptotic solutions is likewise posed; however, the relations for determining the arbitrary functions obtained in paper ([4]) are more complicated than the relations obtained above.

In contrast to paper ([4]) and the present paper, in papers ([5,6]) the problem of directly determining asymptotic solutions is not posed. The aim of papers ([5,6]) is to determine the bounds between which the solution oscillates, and to determine the instantaneous period of oscillations; moreover, these quantities are determined directly through the functions $f(\tau,y)$ and $F(\tau,y)$ entering into equation (1). Such an approach is convenient in those cases where the solution of the "reference" equation cannot be represented in a simple form. However, in those cases where the solution of the "reference" equation can be expressed in terms of special functions, the method of the present paper has certain advantages, since it makes it possible to judge more fully the behavior of the solution and of its derivatives.

In addition, the restrictions $\operatorname{sign} F(\tau,y)=\operatorname{sign} y$ and $\left.\frac{\partial F}{\partial y}\right|_{y=0}\neq 0$, imposed on the function $F(\tau, y)$ in paper ([6]), exclude from consideration a number of cases that can be treated by the method of the present paper.

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Note: Figure translations are in progress. See original paper for figures.