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Abstract

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MATHEMATICS

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AN EXAMPLE OF A ZERO-DIMENSIONAL NORMAL SPACE HAVING INFINITE DIMENSION IN THE SENSE OF COVERINGS

(Presented by Academician P. S. Aleksandrov on 9 VI 1958)

Dowker ([1]) constructed a remarkable normal space M with dimensions* $\operatorname{ind} M = 0$ and $\dim M = 1$. It is topologically embedded in the zero-dimensional (in all senses) bicompactum D^τ, thereby disproving P. S. Aleksandrov' s hypothesis on the monotonicity of the dimension dim in the class of normal spaces.

The aim of this paper is the following generalization of Dowker' s construction, leading to the example indicated in the title (see item H).

A. Let T be the space of all ordinal numbers α not exceeding ω_1, and $W = T \setminus \omega_1$. For each $\alpha < \omega_1$ we shall call the set $W_\alpha = \delta\{\beta : \beta > \alpha\}$ an α-tail. Let P be an arbitrary metric space with a countable base.

We shall call a set M of the product $W \times P$ *convergent* (to P) if for every point $p \in P$ there is a tail W_α such that $W_\alpha \times p \subseteq M$.

Theorem 1. *Every convergent set M of the product $W \times P$ is normal, countably paracompact**, and $\dim M = \dim P$; if, moreover, $\operatorname{ind} P = 0$, then $\dim M = \dim P$ also in the infinite-dimensional cases***.*

To prove this, a series of auxiliary lemmas is needed.

B. *The intersection of a countable or finite number of closed cofinal ω_1-sets A_i of the space W is nonempty****.*

C. *If the sets Γ_i are open in W and $W = \bigcup_i \Gamma_i$, $i = 1, 2, ...$, then at least one of them contains some α-tail W_α.*

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* "Zero-dimensionality" is here to be understood only in the sense of the "small" inductive dimension ind (induction is carried out over points). We shall also

consider the "large" inductive dimension Ind (induction is carried out over closed sets) and, mainly, the dimension dim, defined by means of finite open covers.

** A space R is countably paracompact and normal if into every countable or finite open cover $\{\Gamma_i\}$ one can inscribe a closed cover $\{\Phi_i\}$ such that $\Phi_i \subseteq \Gamma_i$ ([2]).

*** From the point of view of essential mappings there are three types of infinite-dimensional spaces. We shall write $\dim R < \mathfrak{n}$ (respectively, $\dim R < \mathfrak{N}$) if for every countable system π of pairs of closed sets A_i, B_i, $A_i \cap B_i = \emptyset$, there exist closed sets C_i separating the space R between A_i and B_i and such that $\bigcap C_i = \emptyset$ for all i (respectively, for $i \leq N = N(\pi)$). For countably paracompact (respectively, normal) spaces the assertion $\dim R < \mathfrak{n}$ (respectively, $\dim R < \mathfrak{N}$) is equivalent to the following: for any sequence of continuous (real-valued) functions g_i and any number $\varepsilon, \varepsilon > 0$, one can find continuous functions f_i such that $|g_i - f_i| < \varepsilon$ and $\bigcap f_i^{-1}(0) = \emptyset$ (respectively, for $i \leq N(\pi)$), where $\emptyset$ is the empty set.

**** By virtue of cofinality there is a sequence $\{\alpha_i\}$, $\alpha_i < \omega_1$, and a limit ordinal α such that $\alpha_{2^n+i} \in A_i$ for $i = 0, \dots, 2^{n+1} - 1$, and hence $\alpha \in \bigcap_i A_i$.

D. *If the sets Γ_i are open in $W \times P$ and $M \subseteq \bigcup_i \Gamma_i$, $i = 1, 2, \ldots$, then for any point $p \in P$ there exist a neighborhood Op, a tail W_β, and a set Γ_i such that $W_\beta \times Op \subseteq \Gamma_i$.*

Indeed, for the point p there is a tail W_α such that $W_\alpha \times p \subseteq M$. By C there exist a number $\beta \geq \alpha$ and a set Γ_i such that $W_\beta \times p \subseteq \Gamma_i$. For each λ, $\lambda \geq \beta$, take the real number ε_λ—the largest of all numbers $\varepsilon > 0$ such that $\lambda \times O_\varepsilon p \subseteq \Gamma_i$. The number $\delta = \inf_\lambda \varepsilon_\lambda$ is positive, since otherwise there would be order numbers λ_k, converging to a number $\lambda < \omega_1$, such that $\varepsilon_{\lambda_k} \to 0$, whence it would follow that $(\lambda, p) \notin \Gamma_i$ and $\lambda \geq \beta$, contrary to the inclusion $W_\beta \times p \subseteq \Gamma_i$. Thus, $\lambda \times O_\delta p \subseteq \Gamma_i$ for every $\lambda \geq \beta$, which was required to be proved.

E. *If the sets Γ_i are open in $W \times P$ and $M \subseteq \bigcup_i \Gamma_i$, where $i = 1, 2, \ldots$, then there exist an order number β, $\beta < \omega_1$, and open sets U_i of the space P such that $P = \bigcup_i U_i$ and $W_\beta \times U_i \subseteq \Gamma_i$ for each i.*

Indeed, by D, for any point $p \in P$ there exist numbers $\beta(p) < \omega_1$ and $i(p) < \omega_0$ and a neighborhood Op such that $W_{\beta(p)} \times Op \subseteq \Gamma_{i(p)}$. Choose a countable number of sets Op_j covering the space P, with countable base. Let $\beta = \sup \beta(p_j)$, and let

$$U_i = \bigcup_{i(p_j)=i} Op_j$$

for each i. This is what is needed.

F. *Every continuous function f, defined on M, is finally constant, i.e. there is a number β, $\beta < \omega_1$, such that if $\beta \leq \gamma < \gamma' < \omega_1$ and $(\gamma, p) \in M$, then $f(\gamma, p) = f(\gamma', p)$.*

Indeed, let $p \in P$. Then there is a number $\beta_p < \omega_1$ such that, if $\gamma \geq \beta_p$, then $(\gamma, p) \in M$ and $f(\gamma, p) = f(\beta_p, p)$ (see ([3]), p. 300). Take a countable dense set of points p_k from P and the number $\beta = \sup \beta_k$. Then, if $\gamma \geq \beta$, then $(\gamma, p_k) \in M$ and $f(\gamma, p_k) = f(\beta, p_k)$. If $p \neq p_k$, but $\beta \leq \gamma < \gamma' < \omega_1$ and $(\gamma, p) \in M$, then let $p_{k_j} \to p$. Then we have: $f(\gamma', p) = \lim_j f(\gamma', p_{k_j}) = \lim_j f(\gamma, p_{k_j}) = f(\gamma, p)$, which was required to be proved.

Proof of the theorem. We shall use the definition of dimension dim by means of "partitions" *. Let $\dim P < m$, where $m \in \{0, 1, 2, \dots, \mathfrak{w}, \mathfrak{W}\}$. We shall prove that $\dim P < m$. Let A_i, B_i, $A_i \cap B_i = \emptyset$, $i = 1, 2, \dots$, be closed subsets of M. There exist sets Γ^i_A, Γ^i_B, open in $W \times P$, such that

$$M \cap \Gamma^i_A = M \setminus A_i \quad \text{and} \quad M \cap \Gamma^i_B = M \setminus B_i.$$

Applying Lemma E to each pair $\{\Gamma^i_A, \Gamma^i_B\}$ and taking a sufficiently large number β, $\beta < \omega_1$, we find that for the closed subsets $A'_i = P \setminus U^i_A$, $B'_i = P \setminus U^i_B$ of P the relations

$$A_i \cap (W_\beta \times P) \subseteq W_\beta \times A'_i, \qquad B_i \cap (W_\beta \times P) \subseteq W_\beta \times B'_i$$

hold, and $A'_i \cap B'_i = \emptyset$. There are closed sets C'_i, partitioning the space P between A'_i and B'_i, the required number of which have empty intersection. The sets $W_\beta \times C'_i$ partition $W_\beta \times P$ and, in the required number, have empty intersection. The product $\mathcal{E}\{\alpha : \alpha \leq \beta\} \times P$ has a countable base and, hence, $\dim(\mathcal{E}\{\alpha : \alpha \leq \beta\} \times P) < m$**. Hence, in the finite case, and under the condition $\operatorname{ind} P = 0$, also in the infinite case, we obtain that

$$\dim(M \setminus (W_\beta \times P)) = \operatorname{ind} P < m.$$

Since $M \setminus (W_\beta \times P)$ is open and closed in M, there exist sets C''_i, closed in M, complementing $M \setminus (W_\beta \times P)$,

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* $\dim R < n$, if for every system of $(n+1)$ pairs $\{A_i, B_i\}$ of closed sets A_i, B_i, $A_i \cap B_i = \emptyset$, one can find closed sets C_i, partitioning the space R between A_i and B_i, such that $\bigcap C_i = \emptyset$. The equivalence of this definition with the usual one for normal spaces was proved by P. S. Aleksandrov.

** This is also true in the infinite-dimensional case, but it obviously does not follow from this that $\dim(M \cap (\mathcal{E}\{\alpha : \alpha \leq \beta\} \times P)) < m$. For this it is sufficient to require, for $\mathfrak{w}$, that $\operatorname{ind} P < \omega_0$, and, for $\mathfrak{W}$, that $\operatorname{ind} P < \omega_2$.

separating it between $A_i \cap M \setminus (W_\beta \times P)$ and $B_i \cap M \setminus (W_\beta \times P)$ with empty intersection in the required number. Then the sets $C'_i \cup (W_\beta \times C'_i)$ separate M between A_i and B_i and, in the required number, give an empty intersection, as was to be proved. Normality and countable paracompactness are proved analogously.

Conversely, let $A'_i, B'_i,\ A'_i \cap B'_i = \emptyset$, be closed sets in P. If $\dim M < m$, then there exist continuous functions f_i, defined on M, such that

$$f_i(M \cap (W \times A'_i)) = 1, \quad f_i(M \cap (W \times B'_i)) = -1 \quad \text{and} \quad \bigcap f_i^{-1}(0) = \emptyset$$

for the required number of functions. By virtue of F one may put $f'_i(p) = f_i(\gamma_p, p)$, where γ_p is sufficiently large, and prove the continuity of the functions f'_i. We have

$$f'_i(A'_i) = 1, \quad f'_i(B'_i) = -1 \quad \text{and} \quad \bigcap f_{i'}^{-1}(0) = \emptyset$$

for the required number of functions f'_i, i.e. $\dim P < m$, as was to be proved.

Theorem 2. *For every metric space P with a countable base, in the product $W \times P$ there exists a zero-dimensional (i.e.* $\operatorname{ind} M = 0$*) convergent set M (of the same dimension* dim *as P).*

G. *Every metric space with a countable base can be represented as the sum of zero-dimensional sets forming a nondecreasing transfinite sequence of type ω_1.* *

Indeed, for the interval E^1 such a representation was obtained by Dowker **(1). Let**

$$E^1 = \bigcup_\alpha P^1_\alpha, \quad \alpha < \omega_1, \quad P^1_\alpha \subset P^1_{\alpha'} \quad \textbf{for} \quad \alpha < \alpha'.$$

For the Hilbert cube $E^\infty = (E^1)^{\aleph_0}$ an analogous representation

$$E^\infty = \bigcup_\alpha P^\infty_\alpha$$

is made up of powers * of the sets P^1_α:

$$P^\infty_\alpha = (P^1_\alpha)^{\aleph_0}.$$

For an arbitrary space P with a countable base, the required representation

$$P = \bigcup_\alpha P_\alpha$$

is obtained by intersection: if $P \subset E^\infty$, then $P \cap P^\infty_\alpha = P_\alpha$.

Proof of the theorem. Let

$$P = \bigcup_\alpha P_\alpha$$

be the representation obtained in G. The set

$$M = \bigcup_\alpha (W_\alpha \times P_\alpha)$$

is convergent and $\operatorname{ind} M = 0$, as was to be proved.

H. The desired example is obtained by taking the representation of the Hilbert cube E^∞ constructed in G and the corresponding convergent set M of the product $W \times E^\infty$.

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* Under the assumption of the continuum hypothesis this is obvious.

** Let Q_0 be the set of rational points of the interval $E^1 = [0; 1]$, and let Q_α be the congruence classes modulo Q_α, numbered by ordinal numbers α, $0 \leq \alpha < \omega(c)$. The sets

$$P^1_\alpha = \bigcup_{\gamma \geq \omega_1} Q_\gamma \cup \bigcup_{\beta < \alpha} Q_\beta$$

are zero-dimensional for $\alpha < \omega_1$, and, moreover,

$$E^1 = \bigcup P^1_\alpha.$$

*** That is, the Tychonoff product of the spaces P^1_α with themselves, taken the required number of times. This construction was suggested to me by S. Proizvolov. Obviously, $\operatorname{ind} P^\infty_\alpha = 0$; if $\alpha < \alpha'$, then $P^\infty_\alpha \subset P^\infty_{\alpha'}$; if $\{x_i\} \in E^\infty$, then, taking numbers α_i such that $x_i \in P^1_{\alpha_i}$, we find that $\{x_i\} \in P^\infty_\alpha$ for $\alpha = \sup \alpha_i$.

Note: Figure translations are in progress. See original paper for figures.