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Abstract

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MATHEMATICS

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AN EXPANSION FORMULA FOR AN ARBITRARY FUNCTION IN A SERIES IN THE FUNDAMENTAL FUNCTIONS OF A CLASS OF BOUNDARY-VALUE PROBLEMS WITH A PARAMETER FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev on 20 XI 1957)

In this note an expansion formula is given for an arbitrary function from L_2 in a series in the fundamental functions of a class of boundary-value problems with a parameter for linear partial differential equations (see formula (12)). The class of problems under consideration is characterized by the fact that in it the variables are partially separated, and for boundary-value problems obtained by partial separation of variables an expansion formula of type (12) is valid.

Formula (12) generalizes the well-known formula of Ya. D. Tamarkin for ordinary linear differential equations (1) to the case of the class of boundary-value problems considered for equations with partial derivatives. On the basis of this formula, by the method of (2) one can also show that a sufficiently smooth solution of the corresponding mixed problem is represented in the form of an integral residue. Mixed problems of this kind, for which the spectral problems do not correspond to self-adjoint operators, occur, for example, in subsurface hydromechanics and in the theory of heat conduction.

Let us have the equation

$$\mathcal{L}_1\left(\tilde{x}, \frac{\partial}{\partial \tilde{x}}, \lambda\right) v + a(\tilde{x}) \mathcal{L}_2\left(\tilde{\tilde{x}}, \frac{\partial}{\partial \tilde{\tilde{x}}}\right) v = f(x), \tag{1}$$

considered in a bounded domain D of points $x = (x_1, \dots, x_n)$ of Euclidean space, which is the Cartesian product of domains D_1, D_2, described, respectively, by the points $\tilde{x} = (x_1, \dots, x_s)$, $\tilde{\tilde{x}} = (x_{s+1}, \dots, x_n)$, where $\mathcal{L}_1, \mathcal{L}_2$ are linear differential operators respectively in $\tilde{x}$ and $\tilde{\tilde{x}}$, and $\mathcal{L}_1$ has the form:

$$\mathcal{L}_1\left(\tilde{x}, \frac{\partial}{\partial \tilde{x}}, \lambda\right) = \sum_{\substack{mk+l\leq p \\ k\leq q-1}} \lambda^{mk} A_{kl_1\dots l_s}(\tilde{x}) \frac{\partial^l}{\partial x_1^{l_1} \cdots \partial x_s^{l_s}} - \lambda^p;$$

m, q are natural numbers such that $p = mq$. We denote the boundary of the domain D_i by Γ_i $(i = 1, 2)$.

We shall now consider the boundary-value problem of finding a solution $v(x, f, \lambda)$ of equation (1) under the boundary conditions

$$\lim_{\tilde{x}\to\tilde{y}} \sum_{k=0}^{q} \lambda^{mk} B_k\left(\tilde{y}, \frac{\partial}{\partial \tilde{x}}\right) v(x) = 0, \quad \tilde{y} \in \Gamma_1; \tag{2}$$

$$\lim_{\tilde{x}\to\tilde{y}} C\left(\tilde{y}, \frac{\partial}{\partial \tilde{x}}\right) v(x) = 0, \quad \tilde{y} \in \Gamma_2, \tag{3}$$

where $B_k\left(\tilde{y}, \frac{\partial}{\partial \tilde{x}}\right)$, $C\left(\tilde{y}, \frac{\partial}{\partial \tilde{x}}\right)$ are linear differential operators, respectively, in $\tilde{x}$ and $\tilde{\tilde{x}}$, whose coefficients depend on $\tilde{y} \in \Gamma_1$ and $\tilde{y} \in \Gamma_2$.

Suppose the following conditions are satisfied:

1. Problem (1)–(3) has a unique solution $v(x, f, \lambda)$, analytic in λ for every $f \in L_2(D)$ and complex λ, except for a countable set of values that are poles of this solution.
2. The boundary-value problem of finding a solution z of the equation

$$\mathcal{L}_2\left(\tilde{x}, \frac{\partial}{\partial \tilde{x}}\right) z - \mu z = \varphi(\tilde{x}) \tag{4}$$

under the boundary condition (3) has a unique solution $z(\tilde{x}, \varphi, \mu)$, analytic in μ for every $\varphi \in L_2(D_2)$ and complex μ, except for a countable set of values μ_ν $(\nu = 1, 2, \dots)$, which are poles of this solution; and if $G_2(\tilde{x}, \tilde{\xi}, \mu)$ is the Green’ s function of this problem, then the function $\varphi(\tilde{x})$ can be expanded in a series by the formula*

$$\varphi(\tilde{x}) = -\frac{1}{2\pi\sqrt{-1}} \sum_{\nu} \int_{c_\nu} d\mu \int_{D_2} G_2(\tilde{x}, \tilde{\xi}, \mu) \varphi(\tilde{\xi})\, d\tilde{\xi}, \tag{5}$$

where c_ν is a simple closed contour in the λ-plane enclosing only one pole μ_ν of the function $G_2(\tilde{x}, \tilde{\xi}, \mu)$, and the sum over ν extends over all poles of this function.

3. The problem of finding a solution w of the equation

$$\mathcal{L}_1\left(\tilde{x}, \frac{\partial}{\partial \tilde{x}}, \lambda\right) w + a(\tilde{x})\mu_\nu w = f(\tilde{x}) \tag{6}$$

under condition (2), for $f(\tilde{x}) \in L_2(D_1)$, has a unique solution $w(x, f, \mu_\nu, \lambda)$, analytic in λ for every complex λ, except for a countable set of values $\lambda_{\nu k}$ $(k = 1, 2, ...)$, corresponding to the pole μ_ν $(\nu = 1, 2, ...)$ of the function G_2 and being poles of this solution $w(x, f, \mu_\nu, \lambda)$. Further, if $G_1(\tilde{x}, \tilde{\xi}, \mu_\nu, \lambda)$ is the Green' s function of this problem, then $f(\tilde{x})$ can be expanded in a series by the formula

$$-\frac{1}{2\pi\sqrt{-1}} \sum_k \int_{d_{\nu k}} \lambda^{m(s+1)-1}\, d\lambda \int_{D_1} G_1(\tilde{x}, \tilde{\xi}, \mu_\nu, \lambda) f(\tilde{\xi})\, d\tilde{\xi} = \begin{cases} 0, & \text{for } s < q-1, \\ f(\tilde{x}), & \text{for } s = q-1, \end{cases} \quad (\nu = 1, 2, ...) \tag{7}$$

where $d_{\nu k}$ is a simple closed contour in the λ-plane enclosing only one pole $\lambda_{\nu k}$ of the function G_1, and the sum over k extends over all poles of this function.

* For definiteness, convergence of the series occurring here will be understood in the sense of L_2.

Theorem 1. Under conditions 1-3, if all the poles μ_ν $(\nu = 1, 2, ...)$ of the function $G_2(\tilde{x}, \tilde{\xi}, \mu)$ are simple and if, for some λ and $f \in L_2(D)$, problem (1)–(3) has a sufficiently smooth solution $v(x, f, \lambda)$, then it is representable by the formula

$$v(x, f, \lambda) = -\frac{1}{2\pi\sqrt{-1}} \sum_{\nu=1}^{\infty} \int_{c_\nu} d\mu \int_{\tilde{D}2} G_2(\tilde{x}, \tilde{\xi}, \mu) \int_{\tilde{D}1} G_1(\tilde{x}, \tilde{\xi}, \mu_\nu, \lambda) f(\xi)\, d\xi; \tag{8}$$

and $\lambda \neq \lambda_{\nu k}$.

Proof. Let $F \in L_2(D_2)$. Introduce the notation

$$G_{\nu j}(F) \equiv F_{\nu j}(x) = -\frac{1}{2\pi\sqrt{-1}} \int_{c_\nu} \mu^j\, d\mu \int_{D_2} G_2(\tilde{x}, \tilde{\xi}, \mu) F(\tilde{\xi})\, d\tilde{\xi}.$$

Let $v(x, f, \lambda)$ be a solution of problem (1)–(3). Applying the operator $G_{\nu 0}$ to both sides of (1), (2), we arrive at the identities

$$\mathcal{L}_1\left(\tilde{x}, \frac{\partial}{\partial \tilde{x}}, \lambda\right) v_{\nu 0} + a(\tilde{x}) v_{\nu 1} \equiv f_{\nu 0}(x), \tag{9}$$

$$\lim_{\tilde{x} \to \tilde{y}} \sum_{k=0}^{q} \lambda^{mk} B_k\left(\tilde{y}, \frac{\partial}{\partial \tilde{x}}\right) v_{\nu 0} \equiv 0, \qquad \tilde{y} \in \Gamma_1. \tag{10}$$

In view of the fact that all the poles μ_ν are simple, we have $v_{\nu 1} = \mu_\nu v_{\nu 0}$. Consequently, from the identities (9)–(10) we conclude that $\lambda \neq \lambda_{\nu k}$ $(k = 1, 2, \ldots)$ and

$$v_{\nu 0}(x, f, \lambda) = -\frac{1}{2\pi\sqrt{-1}} \int_{c_\nu} d\mu \int_{D_2} G_2(\tilde{x}, \tilde{\xi}, \mu) \int_{D_1} G_1(\tilde{x}, \tilde{\xi}, \mu_\nu, \lambda) f(\xi)\, d\xi. \tag{11}$$

According to condition (5), the assertion of the theorem follows from (11). From Theorem 1 it follows that the numbers $\lambda_{\nu k}$ $(\nu, k = 1, 2, \ldots)$ exhaust all the poles of the function $v(x, f, \lambda)$.

Now the following theorem is easily proved.

Theorem 2. Under the conditions of Theorem 1, for every function $f \in L_2(D)$ the formula

$$-\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{d_{ij}} \lambda^{m(s+1)-1} v(x, \lambda)\, d\lambda = \begin{cases} 0, & \text{for } s < q - 1, \\ f(x), & \text{for } s = q - 1. \end{cases} \tag{12}$$

holds.

Proof. According to Theorem 1, we have

$$\int_{d_{ij}} \lambda^{m(s+1)-1} v(x, f, \lambda)\, d\lambda =$$

$$= -\frac{1}{2\pi\sqrt{-1}} \int_{d_{ij}} \lambda^{m(s+1)-1}\, d\lambda \int_{D_2} G_2(\tilde{x}, \tilde{\xi}, \mu)\, d\tilde{\xi} \int_{D_1} G_1(\tilde{x}, \tilde{\xi}, \mu_i, \lambda) f(\xi)\, d\tilde{\xi},$$

whence we obtain

$$-\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{d_{ij}} \lambda^{m(s+1)-1} v(x, f, \lambda)\, d\lambda =$$

$$= -\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{\infty} \int_{c_i} d\mu \int_{\tilde{D}2} G_2(\tilde{x}, \tilde{\xi}, \mu)\, d\tilde{\xi} \times$$

$$\times \left\{ -\frac{1}{2\pi\sqrt{-1}} \sum_{j=1}^{\infty} \int_{d_{ij}} \lambda^{m(s+1)-1}\, d\lambda \int_{\tilde{D}_1} G_2(\tilde{x}, \tilde{\xi}, \mu_i, \lambda) f(\xi)\, d\tilde{\xi} \right\}. \tag{13}$$

According to condition 3 (see formula (7)), the expression in braces on the right-hand side of (13) is equal to zero for $s < q-1$, and to $f(\tilde{x}, \tilde{\xi})$ for $s = q-1$. Consequently, from (13), in accordance with condition 2 (see formula (5)), the validity of formula (12) follows.

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CITED LITERATURE

[1] J. Tamarkin, *Math. Zs.*, **27** (1928). [2] M. L. Rasulov, *Matem. sborn.*, **30** (72), issue 3 (1952).

Note: Figure translations are in progress. See original paper for figures.

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