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Abstract

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BICOMPACT EXTENSIONS OF SEMIBICOMPACT SPACES

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A topological space* is called **semibicompact** if it has a base of open sets each of which has a bicompact boundary. Freudenthal (^{1,2}) showed that *the condition of semibicompactness is necessary and sufficient in order that a space R with a countable base possess a bicompact (with a countable base) extension $\widetilde{R}$ with a zero-dimensional** remainder $\widetilde{R} \setminus R$.* In the general case of completely regular spaces this assertion turned out to be false (see (^{7})).

In the present paper, following Freudenthal, bicompact extensions with zero-dimensional remainders are constructed, by means of special bases for semibicompact spaces, using methods of the theory of proximity spaces (Theorems 1, 3, and 5). From these bases, maximal bases in a certain sense are singled out, corresponding one-to-one and isomorphically*** to the constructed bicompact extensions (Theorems 2 and 4). For spaces of a certain class $\mathfrak{C}$, including all metrizable spaces, an analogue of Freudenthal' s general assertion is proved (we take the dimension Ind)—Theorem 7.

§ 1. We shall agree, in semibicompact spaces, to consider only bases consisting of open sets with bicompact boundaries. For each such base $\mathfrak{U}$ we denote by $\widetilde{\mathfrak{U}}$ the minimal**** of all extensions of the base $\mathfrak{U}$ having the property:

a) The extensions $\mathfrak{U}'$ of the base $\mathfrak{U}$, together with arbitrary sets $U_1, \dots, U_k$, contain the complements $R \setminus [U_i]$ and the sum $\bigcup_i U_i$.

Theorem 1. Every base $\mathfrak{U}$ of a semibicompact space R transforms it in the following way into a proximity space (^{3,4}) $R_{\mathfrak{U}}$, compatible with the given topology*****.

Δ) Sets A and B of the space $R_{\mathfrak{U}}$ are considered close ($A \Delta B$), if there is no set $U \in \widetilde{\mathfrak{U}}$ such that $[A] \subseteq U$, while $[B] \subseteq R \setminus [U]$.

Proof. Only axiom B5 requires serious verification (see (^{4}), p. 546). By the following device it is easy to prove that the space

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* By a space we shall everywhere below mean a Hausdorff space, and by a semibicompact space—a non-bicompact semibicompact space.

** Here by dimension is meant the small inductive dimension ind (induction is taken over points). If induction is taken over closed sets, then we obtain the large inductive dimension Ind.

*** In the sense of the natural partial ordering (for bases: $\mathfrak{U} \leq \mathfrak{U}'$, if $\mathfrak{U} \subseteq \mathfrak{U}'$; for bicompact extensions: $\alpha R \leq \gamma R$, if there exists a continuous mapping of the bicompactum γR onto αR, identical on R).

**** By adjoining countably many times, alternately, to the base $\mathfrak{U}$ the complements $R \setminus [U]$ and finite sums, we obtain the minimal base $\widetilde{\mathfrak{U}}$.

***** That is, the assertions $x \in [A]$ and $x \Delta A$ are equivalent ($[A]$ is the closure of the set A in R).

R is regular. Now let $A \bar{\Delta} B$, i.e., there exists a set $U \in \mathfrak{U}$ such that $[A] \subseteq U$ and $[B] \subseteq R \setminus [U]$. For each point x of the boundary $|U|$, by regularity, choose a neighborhood $U_x \in \mathfrak{U}$ such that $[U_x] \subseteq R \setminus [A \cup B]$. From the system of sets U_x choose a finite subsystem $U_1, \dots, U_k$ covering the bicompact $|U|$. Let

$$U_A = R \setminus [(R \setminus U) \cup \bigcup_i U_i], \quad U_B = R \setminus [U \cup \bigcup_i U_i].$$

Then we have: $[B] \subseteq U_B \in \widetilde{\mathfrak{U}}$ and $[U] \cap [U_B] = \Lambda$. Hence, $B \bar{\Delta} [U]$. Similarly, we obtain $A \bar{\Delta} [R \setminus [U]]$, which was required to be proved. •

Corollary. *Every semicompact space is completely regular.*

A. By the theorem of Yu. M. Smirnov (see (4), Chs. 2 and 3), there exists, and moreover only one, bicompact extension $\mathfrak{u}R = \mathfrak{u}R_{\mathfrak{u}}$ of the space R, generating the proximity space $R_{\mathfrak{u}}$:

*$A \Delta B$ if and only if their closures $\mathfrak{u}R[A]$ and $\mathfrak{u}R[B]$ in the bicompactum $\mathfrak{u}R$ intersect.**

B. *If in a semicompact space R bases $\mathfrak{u}, \mathfrak{u}'$ are given and $\mathfrak{u} \subseteq \mathfrak{u}'$, then $R_{\mathfrak{u}} \leqslant R_{\mathfrak{u}'}$** and hence $\mathfrak{u}R \leqslant \mathfrak{u}'R$ ((4), p. 557).*

The converse assertion is false.

Theorem 2. *Among all bases $\mathfrak{u}$ generating one and the same bicompact extension $\widetilde{R} = \mathfrak{u}R$ of a semicompact space R, there is a maximal base.*

Proof. Consider in $\widetilde{R}$ all those open sets O for which $\widetilde{R}|O| \subseteq R$. ***The totality $\tilde{\mathfrak{u}}$ of all sets of the form $O \cap R$, by virtue of the following lemma,*** *contains any base $\mathfrak{u}$ generating the extension $\widetilde{R} = \mathfrak{u}R$, and hence is a base. By the same arguments as in the proof of Theorem 1, one can show*****, that the base $\tilde{\mathfrak{u}}$ generates the proximity space $R_{\mathfrak{u}}$, and hence also the extension $\mathfrak{u}R = \widetilde{R}$, which was required to be proved.

Lemma 1. *For any set U of a base $\mathfrak{u}$ of a semicompact space R, in every bicompact extension $\widetilde{R}$, $\widetilde{R} \geqslant \mathfrak{u}R$, the boundary $\widetilde{R}|O\langle U\rangle|$ of the set $O\langle U\rangle$****** coincides with the boundary $|U|$ of the set U in the space R: $\widetilde{R}|O\langle U\rangle| = |U|$.*

Proof. Always $|U| \subseteq \widetilde{R}|O\langle U\rangle|$. Let us prove the reverse inclusion. Let $U \in \mathfrak{u}$, $\xi \in \widetilde{R}|O\langle U\rangle|$, but $\xi \notin |U|$. Since $\xi \notin |U|$, by bicompactness of the set $|U|$ there is a neighborhood $O\langle H\rangle$ of the point ξ such that $\widetilde{R}[O\langle H\rangle] \cap |U| = \Lambda$*******. Hence, $\xi \in O\langle H\rangle$, i.e. $\xi \ni H$, and also $[H] \cap |U| = \Lambda$. Let $H' = H \cap U$, and $H'' = H \setminus [U]$. Since $|R \setminus [U]| \subseteq |U|$ and $[H] \cap |U| = \Lambda$, we have $H' = H \setminus [R \setminus [H]]$. Therefore

$$H = H' \cup H'',$$

and, according to the definition of proximity in $R_{\mathfrak{U}}$ (since $U \in \mathfrak{U}$),

$$H' \,\overline{\Delta}\, H''.$$

Consequently,

$${}_{\mathfrak{U}}R[H'] \cap {}_{\mathfrak{U}}R[H''] = \Lambda.$$

But $\widetilde{R} \geq {}_{\mathfrak{U}}R$, hence also

$$\widetilde{R}[H'] \cap \widetilde{R}[H''] = \Lambda.$$

Since $\xi \in \widetilde{R}[O\langle U\rangle]$, we have $O\langle U\rangle \cap O\langle\Gamma\rangle \neq \Lambda$, if $\xi \in O\langle\Gamma\rangle$. Therefore, whenever $\xi \ni \Gamma$, then $U \cap \Gamma \neq \Lambda$. But $H \in \xi$. Hence $H \cap \Gamma \neq \Lambda$, whenever $\Gamma \in \xi$. Therefore we have

$$H' \cap \Gamma = U \cap H \cap \Gamma \neq \Lambda,$$

if $\Gamma \in \xi$. Thus

$$H' \cap O\langle\Gamma\rangle \neq \Lambda,$$

as soon as $\xi \in O\langle\Gamma\rangle$. Consequently, $\xi \in \widetilde{R}[H']$.

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* The points of the bicompactum $\mathfrak{u}R$ are all possible ends, i.e. maximal centered Δ-systems ξ of sets of the proximity space $R_{\mathfrak{u}}$ (a system ξ is called a Δ-system if for every set $B \in \xi$ there exists a set $A \in \xi$ such that $A\overline{\Delta}(R \setminus B)$). To points of the space R correspond the ends ξ with nonempty intersection.

** This means that in $R_{\mathfrak{u}}$ there are no fewer pairs of close sets than in $R_{\mathfrak{u}'}$.

*** $\widetilde{R}|\Gamma|$ is the boundary of the set Γ in the bicompactum $\widetilde{R}$.

**** Lemma 1 does not depend on Theorem 2.

***** It is easy to see that the base $\tilde{\mathfrak{u}}$ satisfies condition a) and, moreover, contains differences $U' \setminus [U]$ and finite intersections.

****** For every open set Γ of the space R there exists a set $O\langle\Gamma\rangle$, the largest of all such open sets $\tilde{\Gamma}$ of the bicompact extension $\widetilde{R}$, that $R \cap \tilde{\Gamma} = \Gamma$.

******* The set $O\langle\Gamma\rangle$ consists of all such ends ξ of the proximity space R generated by the bicompact extension $\widetilde{R}$, that $\Gamma \in \xi$. The sets $O\langle\Gamma\rangle$ form a base of the bicompactum $\widetilde{R}$ (see (^{4}), pp. 552 and 558).

Since $\widetilde{R}[H'] \cap \widetilde{R}[H''] = \Lambda$ and $\xi \in \widetilde{R}[H']$, there exists a neighborhood $O\langle W\rangle$ of the point ξ such that

$$\widetilde{R}[O\langle W\rangle] \cap \widetilde{R}[H''] = \Lambda.$$

Hence $\xi \ni W$ and

$$[W] \cap [H''] = \Lambda.$$

Finally, for the set $W \cap H$ we obtain

$$W \cap H \subset H' \subset U$$

and $W \cap H \in \xi$. Therefore $U \in \xi$ and $\xi \in O\langle U\rangle$. Consequently,

$$\xi \notin \widetilde{R}\,|\,\overline{O\langle U\rangle}\,|,$$

as was required to prove.

From the lemma the following theorems follow easily:

Theorem 3. For every bicompact extension ${}_{\mathfrak{U}}R$ (generated by a base $\mathfrak{U}$) of a semibicompact space R, the remainder ${}_{\mathfrak{U}}R \setminus R$ is zero-dimensional*:

$$\operatorname{ind}({}_{\mathfrak{U}}R \setminus R) = 0.$$

Theorem 4. The correspondence between bicompact extensions of the form ${}_{\mathfrak{U}}R$ of a semibicompact space R and its maximal bases $\widetilde{\mathfrak{U}}$ is one-to-one and isomorphic, i.e. the relations

$$\widetilde{\mathfrak{U}} \subset \widetilde{\mathfrak{U}}' \quad \text{and} \quad {}_{\mathfrak{U}}R < {}_{\mathfrak{U}'}R$$

are equivalent.

§ 2. We shall say that a space R has **countable weight on a set** A if there exists a countable system of neighborhoods V_{iA} of the set A such that in every neighborhood UA there is contained one of the neighborhoods V_{iA}. Let $\mathfrak{S}$ be the class of all spaces having countable weight on every bicompact set; metric spaces belong to the class $\mathfrak{S}$.

Theorem 5. For every bicompact extension ${}_{\mathfrak{U}}R$ of a semibicompact space R of class $\mathfrak{S}$, we have:

$$\operatorname{Ind}({}_{\mathfrak{U}}R \setminus R) = 0.$$

Lemma 2. For any base $\mathfrak{U}$ of a semibicompact space R, the sets $O\langle U\rangle$, $U \in \widetilde{\mathfrak{U}}$, form a large base** of the bicompactum ${}_{\mathfrak{U}}R$.

Indeed, if A and B, $A \cap B = \Lambda$, are closed sets of the bicompactum ${}_{\mathfrak{U}}R$, then $A \,\overline{\Delta}\, B$, and hence there exist such sets C and D of the space $R_{\mathfrak{U}}$ that

$$C \,\overline{\Delta}\, D, \qquad A \subseteq O\langle C\rangle, \qquad B \subseteq O\langle D\rangle$$

(see (4), p. 552). Then there is a set $U \in \widetilde{\mathfrak{U}}$ such that

$$[C] \subset U, \qquad [D] \subset R \setminus [U].$$

Hence

$$A \subseteq O\langle C\rangle \subseteq O\langle U\rangle, \qquad B \subseteq O\langle R \setminus [U]\rangle,$$

and

$$O\langle U\rangle \cap O\langle R \setminus [U]\rangle = \Lambda,$$

as was required to prove.

Lemma 3. A completely regular space R has countable weight on a bicompactum Φ, $\Phi \subseteq R$, if and only if Φ has type G_δ in some (every) bicompact extension $\widetilde{R}$ of the space R.

The condition of the lemma is sufficient, because the bicompactum Φ has type G_δ in the bicompactum $\widetilde{R}$ then (5) if and only if $\widetilde{R}$ has countable weight on Φ. Suppose now that $\{V_i\}$ is a base of the space R on the bicompactum Φ, $\Phi \subseteq R$, and $\xi \in \widetilde{R} \setminus \Phi$. Take a neighborhood $O\langle U\rangle$ of the point ξ such that

$$\Phi \cap \widetilde{R}[O\langle U\rangle] = \Lambda,$$

and then a set V_i such that

$$V_i \cap \widetilde{R}[O\langle U\rangle] = \Lambda.$$

Then

$$V_i \cap U = \Lambda$$

and

$$O\langle V_i\rangle \cap O\langle U\rangle = \Lambda.$$

Hence $\xi \notin O\langle V_i\rangle$. Consequently,

$$\Phi = \bigcap O\langle V_i\rangle.$$

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* Hence it follows: every semibicompact space possesses a bicompact extension with a zero-dimensional (in the sense of dimension ind) remainder. The converse assertion, as Yu. M. Smirnov showed, is false. Theorem 3 and the corollary of Theorem 1 belong to Freudenthal.

** A base $\mathfrak{U}$ of open sets of a space R is called **large** if for every closed set A and every neighborhood OA of it there exists a set $U \in \mathfrak{U}$ such that

$$A \subseteq U \subseteq OA.$$

Proof of Theorem 5. Let R be semicompact, and let $\widetilde{R}$ be one of its extensions of type uR. Let A and B, $A \cap B = \Lambda$, be closed sets in $\widetilde{R} \setminus R$. The set

$\Phi = \widetilde{R}[A] \cap \widetilde{R}[B]$ is bicompact and $\Phi \subseteq R$. By Lemma 3 there exist open sets V_i in $\widetilde{R}$ such that $\Phi = \bigcap V_i$ and $\widetilde{R}[V_{i+1}] \subseteq V_i$ $(i = 1, 2, 3, \ldots)$. Put $S_i = \widetilde{R}[V_i] \setminus V_{i+1}$. Then

$$(\widetilde{R}[A] \cap S_i) \cap (\widetilde{R}[B] \cap S_i) = \Lambda$$

and

$$S_i \cap S_{i+2} = \Lambda \quad (i = 1, 2, \ldots).$$

By Lemmas 2 and 1 there exist open sets W_i in $\widetilde{R}$, the boundaries of which lie in R, such that

$$\widetilde{R}[A] \cap S_i \subseteq W_i, \qquad \widetilde{R}[B] \cap S_i \cap \widetilde{R}[W_i] = \Lambda$$

and

$$W_i \cap S_{i-2} = W_i \cap S_{i+2} = \Lambda.$$

The system of sets W_i is locally finite in $\widetilde{R} \setminus \Phi$, while the system of sets

$$W_i' = W_i \setminus R$$

is locally finite in $\widetilde{R} \setminus R$, and moreover

$$A \subseteq \bigcup_i W_i', \qquad B \cap \bigcup_i W_i' = \Lambda.$$

The sets W_i', and together with them also $\bigcup_i W_i'$, are both open and closed in $\widetilde{R} \setminus R$. Thus,

$$\operatorname{Ind}(\widetilde{R} \setminus R) = 0,$$

as was required to prove.

Lemma 4. Let R be a regular space and $x \in R$; then, if

$$\operatorname{Ind}(R \setminus x) \leq n,$$

then also

$$\operatorname{Ind} R \leq n.$$

*

Indeed, let A and B, $A \cap B = \Lambda$, be closed sets in R. Let $x \notin A$ and $R' = R \setminus x$. There is a neighborhood Ox such that

$$A \cap R[Ox] = \Lambda.$$

Put

$$OA = R \setminus B \setminus R[Ox].$$

Since $\operatorname{Ind} R' \leq n$, there is in R' a neighborhood UA such that

$$UA \subseteq OA$$

and

$$\operatorname{Ind} R' \mid UA \mid \leq n-1.$$

But UA is open in R and

$$R|UA| = R'|\overline{UA}|.$$

Hence,

$$\operatorname{Ind} R \leq n.$$

Theorem 6. If

$$\operatorname{Ind}(\widetilde{R} \setminus R) = 0$$

for some bicompact extension $\widetilde{R}$ of a space R of class $\mathfrak{C}$, then R is semicompact.

Proof. Let $x \in R$, and let $\widetilde{O}x$ be a neighborhood of the point x in $\widetilde{R}$. Then (by Lemma 4)

$$\operatorname{Ind}((\widetilde{R} \setminus R) \cup x) = 0,$$

and hence there exists an open-and-closed neighborhood V in $(\widetilde{R} \setminus R) \cup x$ such that

$$x \in V \subseteq \widetilde{O}x.$$

The sets V and

$$W = ((\widetilde{R} \setminus R) \cup x) \setminus V$$

are closed in $(\widetilde{R} \setminus R) \cup x$, and

$$V \cap W = \Lambda.$$

Hence the set

$$\Phi = \widetilde{R}[V] \cap \widetilde{R}[W]$$

is bicompact and $\Phi \subseteq R$. Then (by Lemma 3) the complement $\widetilde{R} \setminus \Phi$ has type F_σ and therefore is normal (^{6}). Hence there exists a neighborhood $\widetilde{Q}$ of the set $\widetilde{R}[V] \setminus \Phi$, whose closure (taken in $\widetilde{R} \setminus \Phi$!) does not meet $\widetilde{R}[W] \setminus \Phi$, i.e.

$$\widetilde{R}[V] \setminus \Phi \subseteq \widetilde{Q}$$

and

$$\widetilde{R}[\widetilde{Q}] \cap \widetilde{R}[W] \subseteq \Phi.$$

Hence

$$V \subseteq \widetilde{Q}, \qquad \widetilde{R}[\widetilde{Q}] \cap W \subseteq \Phi$$

and

$$\widetilde{R} - |\widetilde{Q}| \subseteq R.$$

Finally, for the set

$$\widetilde{U} = \widetilde{Q} \cap \widetilde{Q}x$$

we obtain:

$$x \in \widetilde{U}, \qquad \widetilde{R}[\widetilde{U}] \cap V \subseteq \widetilde{O}x$$

and

$$\widetilde{R}[\widetilde{U}] \cap W \subseteq \Phi.$$

Hence

$$R|\widetilde{U}| \cap \widetilde{R}|\widetilde{O}x| = \Lambda,$$

and therefore

$$\widetilde{R}|\widetilde{U}| \subseteq \widetilde{R}|\widetilde{Q}| \subseteq R.$$

Consequently, for the set $U = \widetilde{U} \cap R$, open in R, we have

$$x \in U \subseteq \widetilde{O}x$$

and

$$R|U| = \widetilde{R}|\widetilde{U}|$$

is bicompact, as was required to prove.

Theorem 7. A space R of class $\mathfrak{C}$ has a bicompact extension $\widetilde{R}$ with zero-dimensional, in the sense of the dimension Ind, remainder $\widetilde{R} \setminus R$ if and only if it is semicompact.

Yu. M. Smirnov communicated that, for spaces of class $\mathfrak{C}$, the remainder N in any bicompact extension is finally compact. Therefore the conditions

$$\dim N = 0, \qquad \operatorname{Ind} N = 0, \qquad \operatorname{ind} N = 0$$

are equivalent.

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CITED LITERATURE

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* As Yu. M. Smirnov has shown, the assertion of the lemma for the dimension ind is not true even in the case of normality of the space R.