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Abstract

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MATHEMATICS

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BOUNDARY-VALUE PROBLEMS AND BIORTHOGONAL EXPANSIONS IN BANACH SPACES

(Presented by Academician S. L. Sobolev on 23 XI 1957)

1. The present note is a development of the Weyl–Vishik method of orthogonal projections ([1,2]) for solving linear partial differential equations of elliptic type. The subsequent exposition will be carried out for the example of the Laplace equation; however, a generalization to equations of higher order and to systems with smooth coefficients is possible. This will not be mentioned further. The domain Ω of the m-dimensional Euclidean space E^m is assumed to be finite and to have a sufficiently smooth boundary S. In ([2]) it was established that the space $L_2(\Omega)$ of vector functions with m components from $L_2(\Omega)$ decomposes into the orthogonal sum

$$\mathbf{L}_2 = \Psi^0 \oplus U \oplus Z, \tag{1}$$

where Ψ^0 are vortices with zero flux through the boundary; U are harmonic gradients, and Z are gradients of functions vanishing on the boundary (what is said about Ψ^0 and Z is understood in the generalized sense of S. L. Sobolev).

The expansion (1) made it possible to solve boundary-value problems when a solution in $W_2^{(1)}(\Omega)$ could be obtained. Further, in ([2]) a remark was made concerning the solution of problems in $W_p^{(1)}(\Omega)$, but the results obtained in this direction were not complete. The remarkable theorem of Calderón and Zygmund ([3]) on singular integrals makes it possible to obtain very complete results on the solution of boundary-value problems and on biorthogonal expansions in $L_p(\Omega)$. We note, however, that the expansion (1) was obtained geometrically and then applied to the solution of boundary-value problems; here we first solve the boundary-value problem and, relying on the Calderón–Zygmund theorem, establish the corresponding expansion.

2. Calderón and Zygmund theorem ([3]). A singular integral operator J of the form

$$Jf \equiv \int_\Omega K(P,Q) f(Q)\, dQ \tag{2}$$

is a bounded operator from $L_p(\Omega)$ into $L_p(\Omega)$, if

$$K(P,Q) = |P-Q|^{-m} \omega\left(\frac{P-Q}{|P-Q|}\right) = \frac{\omega(P,\vartheta)}{r^m},$$

$\int_{S_1} \omega(P,\vartheta)\, dS_1 = 0,$ where S_1 is the unit sphere with center at the point P;

$$\omega \in L_p(S_1).$$

(See also ([4]), where an analogous theorem was proved for $p = 2$.)

3. The solution of the classical problem

$$\Delta u = 0, \qquad u|_S = \varphi(S), \tag{3}$$

where φ is a smooth function, as is known, can be obtained in the form $u = \varphi - z$, where φ is a smooth function that is an extension of φ from S to Ω, and z is the solution of the problem

$$\Delta z = \Delta\varphi, \qquad z|_S = 0. \tag{4}$$

The generalized formulation of problems (3) and (4) is as follows: find functions u and z satisfying the conditions

$$(Gu, G\zeta) = 0, \qquad u - \varphi \in \overset{0}{W}{}_p^{(1)}, \qquad \varphi \in W_p^{(1)}(\Omega), \tag{3'}$$

$$(Gz, G\zeta) = (G\varphi, G\zeta), \qquad z \in \overset{0}{W}{}_p^{(1)} \tag{4'}$$

for all $\zeta \in \overset{0}{W}{}_{p'}^{(1)}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, and G is the gradient operator.

The solution of $(4')$ is given by the well-known Green formula

$$z(P) = -\int_\Omega (GK(P,Q), G\varphi(Q))\, dQ. \tag{5}$$

If S is sufficiently smooth, then for the Green function $K(P,Q)$ the estimates

$$K(P,Q) < Cr^{-m+2}, \qquad \left|\frac{\partial K}{\partial x_i}\right| < Cr^{-m+1}, \qquad \left|\frac{\partial^2 K}{\partial x_i \partial x_j}\right| < Cr^{-m} \tag{6}$$

hold.

It is not difficult to show (see, for example, ([5])) the applicability to Gz of the Calderón–Zygmund theorem, which gives us the decomposition

$$\begin{aligned} F_p &= U_p + Z_p \\ F_{p'} &= U_{p'} + Z_{p'} \end{aligned} \tag{7}$$

where F_p is the space of gradients of all functions $\varphi \in W_p^{(1)}$, and the subspaces joined by arrows are orthogonal. Incidentally, let us note that if $\varphi \in W_p^{(2)}$ and Ω is a simply connected domain, then for problem (4) we obtain Koshelev' s result ([6]), while for problem (3) we obtain a stronger one, since from $\varphi \in W_2^p(\Omega)$ it does not follow that $\varphi|_S \in W_p^{(2)}(S)$; this shows the naturalness of specifying not the smoothness of φ as an element of S, but the extendability of φ from S to Ω.

In a completely analogous way we obtain the decomposition

$$\begin{aligned} L_p &= \Psi_p^* + Z_p \\ L_{p'} &= \Psi_{p'} + Z_{p'} \end{aligned} \tag{8}$$

Passing to the second boundary-value problem,

$$\Delta u = 0, \qquad \left.\frac{\partial u}{\partial n}\right|_S = \psi_n|_S, \qquad \int_S \psi_n \, dS = 0, \tag{9}$$

where Ψ_n is the projection on the outward normal of the vector function Ψ given in Ω, we note that speaking of the derivatives of u taking certain values on a manifold of dimension less than m is sometimes possible only in a generalized sense.

in the generalized sense (7). A smooth solution of (9) satisfies the integral identity

$$(Gu,\, G\xi) = (\psi,\, G\xi) \quad \text{for all } \xi \in W_{p'}^{(1)}; \tag{9'}$$

we shall take it as the definition of a generalized solution of the Neumann problem. $\psi \in \Psi_p$, i.e. $\psi \perp Z_{p'}$.

We obtain the solution of $(9')$ by means of the Neumann function in the form

$$u(P) = \int_\Omega (GK(P, Q),\ \psi(Q))\, dQ. \tag{10}$$

By the Calderon–Zygmund theorem we obtain $Gu \in L_p(\Omega)$ and, correspondingly, $\psi^0 \equiv \psi - Gu \in L_p(\Omega)$.

Thus we obtain the decomposition

$$\begin{array}{c} \Psi_p = \Psi_p^0 + U_p \\ \searrow\!\!\nearrow \quad \swarrow\!\!\searrow \\ \Psi_{p'} = \Psi_{p'}^0 + U_{p'}. \end{array} \tag{11}$$

Combining (7) and (11), we finally obtain

$$\begin{aligned} \mathbf{L}_p &= \Psi_p^0 + U_p + Z_p, \\ \mathbf{L}_{p'} &= \Psi_{p'}^0 + U_{p'} + Z_{p'}. \end{aligned} \tag{12}$$

The harmonicity of the functions u is proved as in the book of S. L. Sobolev ([7]).

Returning to the second boundary-value problem, let us note that if ψ belongs in Ω to such a class that the value $\psi_n|_S$ generates a linear functional in $W_p^{(1)}(\Omega)$, then we obtain the adoption of the boundary conditions $\partial u/\partial n$ in a certain weak sense, as indicated in ([7]).

Let us also note that if the right-hand side or the boundary conditions possess generalized derivatives of higher order $\varphi \in W_p^{(l)}(\Omega)$, then the Calderon–Zygmund theorem immediately proves the membership of z and u in $W_p^{(l-2)}(\Omega)$, respectively, and the adoption by the functions $\partial u/\partial n$ of the boundary values in a stronger sense.

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Note: Figure translations are in progress. See original paper for figures.