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Abstract

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THE PRINCIPLE OF MONOTONICITY OF THE ARGUMENT AND DIFFERENTIATION OF INEQUALITIES

(Presented by Academician S. L. Sobolev on 12 I 1958)

1. In paper (^{1}) it was shown that a majorization relation is preserved under differentiation, i.e., from the inequality $|f(x)| \leq |\omega(x)|$ in a neighborhood of a real point x there follows the inequality $|f'(x)| \leq |\omega'(x)|$, if for every φ the argument $\arg[\omega(x) - e^{i\varphi} f(x)]$ in some neighborhood of the point x changes in one and the same direction. We called this condition the **condition of monotonicity of the argument**. Thus the derivation of inequalities analogous to S. N. Bernstein' s inequality is reduced to verifying fulfillment of the condition of monotonicity of the argument. In paper (^{1}) a scheme was given for verifying this condition. This scheme requires the following: in the domain $\mathfrak{G}$, on whose boundary we wish to derive an inequality for derivatives, Lindelöf' s maximum principle must hold; the majorant $\omega(z)$ must satisfy the condition $|\bar{\omega}(z) : \omega(z)| \leq 1$, and the function $f(z)$ the condition of subordination $|f(z) : \omega(z)| \leq 1$ and $|\bar{f}(z) : \omega(z)| \leq 1,\ z \in \mathfrak{G}$. Here $\bar{F}(z)$ is understood to be the function taking on the boundary of the domain $\mathfrak{G}$ values conjugate to the values $F(z)$.

The condition of monotonicity of the argument and the scheme based on this condition made it possible to break away from the class of entire functions of exponential type, to which, before paper (^{1}), all generalizations of S. N. Bernstein' s inequality had been limited, and to transfer this inequality to very general classes of functions ($^{1-4}$). In all the papers listed only single-valued functions were considered, although the scheme does not require this. In the present paper we implement the scheme for multivalued functions.

2. As the domain $\mathfrak{G}$ we shall take, for simplicity of exposition, the open plane $z \neq \infty$ with cuts along a system β of closed intervals of the real axis. The system β consists of a finite or countable number of closed intervals having no common points. The half-infinite intervals $-\infty < x \leq c_1$ and $c_2 \leq x < +\infty$, and the infinite interval $(-\infty, +\infty)$, may be included in the system β. By **regular points** of the boundary β we shall mean the interior points of both shores of the cuts. By $\widetilde{\mathfrak{G}}$ we denote the universal covering surface of the domain $\mathfrak{G}$, by $\tilde{\beta}$ the boundary of $\widetilde{\mathfrak{G}}$ lying over β;

by **regular points** of $\tilde{\beta}$ we shall mean points lying over regular points of β.

It can be shown that in the domain $\mathfrak{G}$ Lindelöf' s principle is valid in the following formulation: let $\varphi(z)$ be a bounded multivalued analytic function in $\mathfrak{G}$, continuous up to β; then $\sup|\varphi(z)|,\ z \in \mathfrak{G}$, does not exceed $\sup|\varphi(x)|,\ x \in \beta$.

A function $f(z)$, single-valued and analytic in $\mathfrak{G}$ ($\widetilde{\mathfrak{G}}$), will be called **admissible** if in $\mathfrak{G}$ ($\widetilde{\mathfrak{G}}$) there exists a single-valued analytic function $\bar{f}(z)$, taking on the boundary β ($\tilde{\beta}$) values conjugate to the values $f(z)$. From the principle of symmetry it follows that an admissible function can be represented in the form $f(z) = g(z) + ih(z)$, where $g(z)$ and $h(z)$ are real on the boundary β ($\tilde{\beta}$), single-valued and analytic in $\mathfrak{G}$ ($\widetilde{\mathfrak{G}}$) up to the regular points of the boundary β ($\tilde{\beta}$), and continuous up to the boundary β ($\tilde{\beta}$).

Definition. The class HB in $\mathfrak{G}$ (or $\widetilde{\mathfrak{G}}$) consists of functions admissible in $\mathfrak{G}$ ($\widetilde{\mathfrak{G}}$)

$$\overline{\omega}(z) = u(z) - iv(z),$$

having no common zeros with the function

$$\omega(z) = u(z) + iv(z)$$

and satisfying in $\mathfrak{G}$ (respectively $\widetilde{\mathfrak{G}}$) the inequality

$$|\overline{\omega}(z) : \omega(z)| < 1. \tag{1}$$

If $\overline{\omega}(z) \in HB$ in $\widetilde{\mathfrak{G}}$, then the function

$$\zeta(z) = \overline{\omega}(z) : \omega(z)$$

maps $\widetilde{\mathfrak{G}}$ many-sheetedly into $|\zeta| < 1$, while $\tilde{\beta}$ is mapped into the circle $|\zeta| = 1$. Therefore $\arg \omega(z)$ strictly decreases along $\tilde{\beta}$, and if t is a parameter increasing monotonically in traversing some component of $\tilde{\beta}$, and at regular points $\dot{z} = dz/dt \neq 0$, then at regular points

$$\dot{v}u - \dot{u}v < 0.$$

It follows from (1) that all zeros of $u(z)$ and $v(z)$ belong to $\beta(\tilde{\beta})$, and from the monotonicity of $\arg \omega(z)$ it follows that the zeros of $u(z)$ alternate with the zeros of $v(z)$. At regular points of $\beta(\tilde{\beta})$, $u(z)$ and $v(z)$ can have only simple zeros, and at singular points only zeros of order 1/2. Since $(a - ib)\overline{\omega}(z) \in HB$, what has been said also applies to the linear combinations $au(z) - bv(z)$ and $bu(z) + av(z)$.

An admissible function $f(z)$ is called **subordinate** to the admissible function $\omega(z)$, if in $\widetilde{\mathfrak{G}}$

$$|\overline{f}(z) : \omega(z)| < 1 \quad \text{and} \quad |f(z) : \omega(z)| < 1.$$

From Lindelöf' s maximum principle it follows:

Theorem 1. *Let the admissible function $f(z)$ be subordinate to the function $\omega(z)$, and let $\overline{\omega}(z) \in HB$. Then for every t, with modulus less than 1, the function*

$$\overline{\omega}_t(z) = \overline{\omega}(z) - t\overline{f}(z)$$

belongs to the class HB in $\widetilde{\mathfrak{G}}$, and $\arg \omega_t(z)$ decreases monotonically along the boundary $\tilde{\beta}$.

Corollary. At regular points of the boundary of $\widetilde{\mathfrak{G}}$, for the functions $\omega(z)$ and $f(z)$ the conditions of monotonicity of the argument are satisfied; therefore at these points all inequalities of Chapter I of work (3) are satisfied. In particular, for any γ and δ, $\operatorname{Im}(\gamma : \delta) \leqslant 0$,

$$|\gamma f'(x) + \delta f(x)| \leqslant |\gamma\omega'(x) + \delta\omega(x)|. \tag{2}$$

If x_1 and x_2 are sufficiently close boundary points, then

$$|f(x_2) - f(x_1)| \leqslant |\omega(x_2) - \omega(x_1)|. \tag{3}$$

Theorem 2. *Let the function $f(z)$ be subordinate to the function $\omega(z)$, $\overline{\omega}(z) \in HB$, and suppose that at some regular point of the boundary of $\widetilde{\mathfrak{G}}$ equality holds in (3) ($\gamma \neq 0$). Then*

$$f(z) = c_1\omega(z) + c_2\overline{\omega}(z), \qquad |c_1| + |c_2| = 1. \tag{4}$$

The converse assertion is also true (with one simple exception). The proof is based on the fact that the function $\omega(z) - e^{i\alpha} f(z)$ cannot have multiple zeros, on Theorem 1.2 of (3), and on the following lemma, which is a certain generalization of a well-known theorem of Iversen.

Lemma. *Let $w(z)$ be a function analytic in $\widetilde{\mathfrak{G}}$, continuous up to $\partial\tilde{\beta}$. Let $\mathfrak{M}$ be the closure of the set of values of $w(z)$ on $\tilde{\beta}$, and let U be some component of the set, complementary in the open plane $w \neq \infty$ to $\mathfrak{M}$. If the point $w_1 = w(z_1)$, $z_1 \in \widetilde{\mathfrak{G}}$, lies in U, and w_0 is any point of U, then $w(z)$ assumes in $\widetilde{\mathfrak{G}}$ values arbitrarily close to w_0. If $w(z)$ does not assume the value w_0 in $\widetilde{\mathfrak{G}}$, then w_0 is an asymptotic value of the function $w(z)$ in $\widetilde{\mathfrak{G}}$.*

3. Let $\overline{\omega}(z) \in HB$ and let $f(z)$ be an admissible function. The least upper bound of the ratios

$$|f(z) : \omega(z)| \quad \text{and} \quad |\overline{f}(z) : \omega(z)|$$

in $\widetilde{\mathfrak{G}}$ is called the **relative deviation** of the function $f(z)$ from $\omega(z)$ in $\widetilde{\mathfrak{G}}$.

It follows from Theorem 2 that:

Theorem 3. The least relative deviation of an admissible function $f(z)$ in $\mathfrak{G}$, normalized by the condition $\gamma f'(\xi)+\delta f(\xi)=1$ $(\gamma\neq 0)$, where ξ is a regular point of $\tilde{\beta}$ relative to the function $\omega(z)$, $\bar{\omega}(z)\in HB$, is equal to $L=|\gamma\omega'(\xi)+\delta\omega(\xi)|^{-1}$ and is attained by any function of the form

$$f(z)=c(\gamma\omega'(\xi)+\delta\omega(\xi))^{-1}\omega(z)+(1-c)(\bar{\gamma}\bar{\omega}'(\xi)+\bar{\delta}\bar{\omega}(\xi))^{-1}\bar{\omega}(z),\quad 0\leq c\leq 1, \tag{5}$$

and only by these functions.

Remark. If the additional requirement of reality of $f(z)$ on $\tilde{\beta}$ is imposed, there exists only one extremal function

$$zf(z)=(\gamma\omega'(\xi)+\delta\omega(\xi))^{-1}\omega(z)+(\bar{\gamma}\bar{\omega}'(\xi)+\bar{\delta}\bar{\omega}(\xi))^{-1}\bar{\omega}(z). \tag{6}$$

There is also a theorem analogous to Theorem II.18 of ([3]).

4. Let $\{a_n\}$ and $\{b_n\}$ be two sequences with the unique possible limit point $z=\infty$, and let all ends of the intervals β belong to the sequence $\{b_n\}$. Construct the functions

$$u(z)=e^{g(z)}\prod\left(1-\frac{z}{a_n}\right)e^{P_n\left(\frac{z}{a_n}\right)},\qquad v(z)=-ie^{h(z)}\prod\left(1-\frac{z}{b_n}\right)^{\delta_n Q_n\left(\frac{z}{b_n}\right)}, \tag{7}$$

where $g(z)$ and $h(z)$ are real entire functions; $P\left(\frac{z}{a_n}\right)$ and $Q\left(\frac{z}{b_n}\right)$ are Weierstrass exponents; $\delta_n=\frac{1}{2}$, if the point b_n is an end of an interval β, and $\delta_n=1$ otherwise.

Theorem 4. Necessary and sufficient conditions for the function $\omega_{\lambda,-\mu}(z)=u(z)-i\mu v(z)$, where λ and μ are real constants, to belong to the class HB in the domain $\mathfrak{G}$ are the following: 1) all zeros a and b of the functions $u(z)$ and $v(z)$ belong to β; 2) on each segment β the zeros of the function $v(z)$ strictly alternate with the zeros of the function $u(z)$; 3) the series convergent by virtue of the preceding conditions

$$h(z)-g(z)+\sum\left(\delta_n Q\left(\frac{z}{b_n}\right)-P\left(\frac{z}{a_n}\right)\right) \tag{8}$$

reduces to a real constant; 4) at least at one point $\xi\in\beta$, $\lambda\mu((\dot{v}u-u\dot{v})<0$.

We note that $\omega_{\lambda,-\mu}(z)=\bar{\omega}_{\lambda,\mu}(z)$.

Example 1. β consists of the segment $[-1, +1]$. As $\omega_{1,1}(z)$ one may take

$$\omega_{1,1}(z) = \cos n \arccos z + i \sin n \arccos z = (z + i\sqrt{1 - z^2})^n.$$

Example 2. β consists of the intervals $-\infty < x \leq -1$ and $1 \leq x < +\infty$. As $\omega_{1,1}(z)$ one may take

$$\omega_{1,1}(z) = \cos\sqrt{z^2 - 1} - i \sin\sqrt{z^2 - 1} = e^{-i\sqrt{z^2-1}}.$$

(see (6)).

5. An interesting construction for building a majorant is given in a paper of Schaeffer ([5]). Unfortunately, the author of that paper was apparently not acquainted with the works ([1–3]), and therefore, in deriving inequalities for derivatives, instead of the condition of monotonicity of the argument, which is fulfilled in this case, he uses the Cauchy integral. In this way the inequality is weakened and is derived only for entire functions. N. I. Akhiezer and B. Ya. Levin in ([6]) freed themselves from one restriction of Schaeffer' s on the boundary of the domain $\mathfrak{G}$ and applied our scheme of majorants, based on the principle of monotonicity of the argument.

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Note: Figure translations are in progress. See original paper for figures.